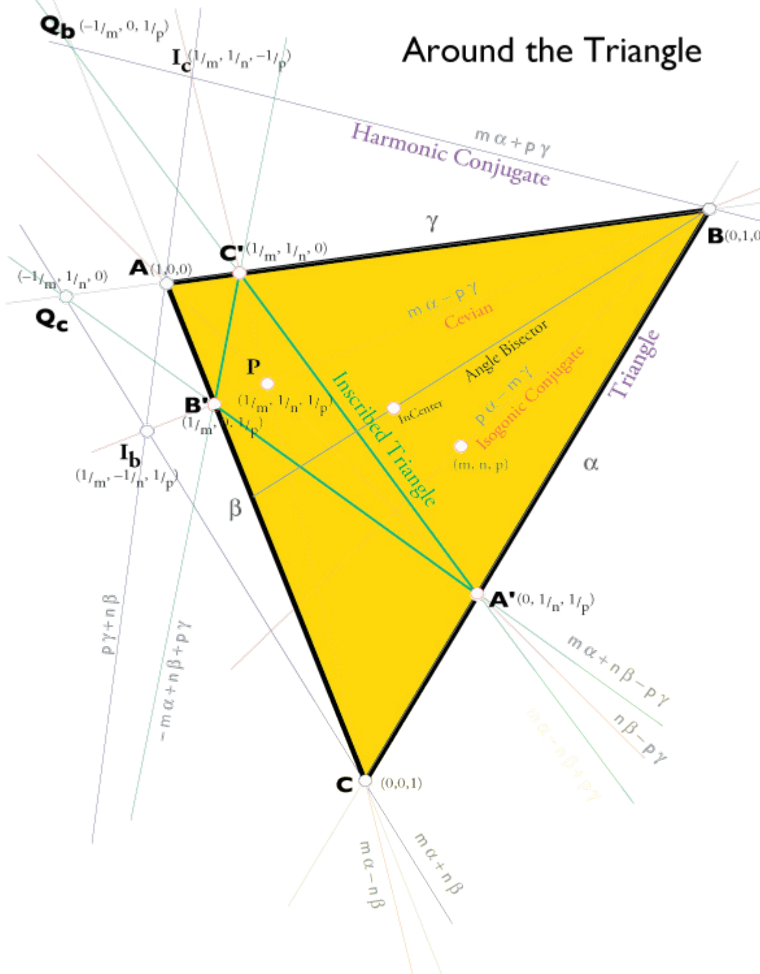
TRILINEAR LINES

Around the Triangle



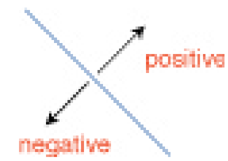
by Steve Sigur

Straight Lines

There is a special way to write straight lines, which I call the normal form, $\mathbf{ax + by + c = 0}$ where $\mathbf{a^2 + b^2 = 1}$.

We get this from the general form $\mathbf{Ax + By + C = 0}$ by dividing by $\sqrt{\mathbf{A^2 + B^2}}$.

This form of the straight line is special because $\mathbf{a\, x_1 + b\, y_1 + c}$ is the signed distance from the point $\mathbf{P(x_1, y_1)}$ to the line $\mathbf{ax + by + c = 0}$. Usually the distance formula from a point to a line has an absolute value operation; leaving off the absolute value in this distance formula, creating a signed distance, is an advantage. This formula now has a dual role, simultaneously being the distance from a line to a point and an equation for a line. The number **c** is the distance from the origin to the line.



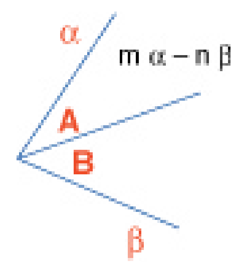
Example: consider the line $\mathbf{x + y + 1 = 0}$. In normal form, this is $\mathbf{x/\sqrt{2} + y/\sqrt{2} + 1/\sqrt{2} = 0}$. The distance from the origin to this line is $\mathbf{1/\sqrt{2}}$. The distance from (1,1) to the line is $3/\sqrt{2}$.

Note that $\mathbf{-ax-by-c=0}$ is the same line as $\mathbf{ax+by+c=0}$, but that the opposite side of the line has positive direction.

Let $\alpha = \mathbf{0}$ and $\beta = \mathbf{0}$ be straight lines in this normalized form. Then $\mathbf{m}\,\alpha + \mathbf{n}\,\beta = \mathbf{0}$, where **m** and **n** are numbers, is another straight line which goes through the intersection point of **a** and **b**. By adjusting **m** and **n**, any straight line through this point can be obtained.

α is a function of **x** and **y**. $\alpha(\mathbf{P})$ is that function evaluated at point **P**. If **P** is on the line, $\alpha(\mathbf{P}) = \mathbf{0}$; otherwise $\alpha(\mathbf{P})$ is the distance from **P** to the line. **a** thus has a dual role, being the equation of a line in some circumstances and the distance from a point to a line in others. This duality both reflects the dual nature of points and lines in triangles as well as gives this method computational power.

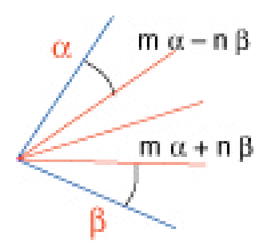
α + **constant** is parallel to α. The constant is the distance between lines.



Special lines related to α and β

$\alpha - \beta = \mathbf{0}$ is the bisector of one of the angles formed by α and β. $\alpha + \beta = \mathbf{0}$ is the bisector of its supplement. This can be shown as follows. Since the bisector goes through the intersection of α and β, it can be written as $\mathbf{m}\,\alpha + \mathbf{n}\,\beta = \mathbf{0}$. Choosing the bisector in the region where both distances are positive we have $\alpha(\mathbf{P}) = \beta(\mathbf{P})$ where P is any point in the bisector. They cancel leaving $\mathbf{m + n = 0}$. We then substitute for **m** in the original expression for the line and cancel the **n**'s obtaining $\alpha - \beta = \mathbf{0}$.

Any line through the point of intersection of lines α and β can be written $\mathbf{m}\,\alpha + \mathbf{n}\,\beta = \mathbf{0}$. **m** and **n** can be written in terms of the angles with which the line intersects α and β (see picture). Letting **P** be on the line we have $\mathbf{m}\,\alpha(\mathbf{P}) + \mathbf{n}\,\beta(\mathbf{P}) = \mathbf{0}$. Since $\alpha(\mathbf{P})$ is the distance from line **a** to point **P**, it can be evaluated as $\mathbf{a(P) = OP \sin A}$. Likewise $\beta(\mathbf{P}) = \mathbf{OP \sin B}$. Substituting we get that $\mathbf{m = \sin B}$ and $\mathbf{n = - \sin A}$. Hence the desired equation of the line is $\mathbf{\sin B} \cdot \alpha - \mathbf{\sin A} \cdot \beta = \mathbf{0}$.



If $\mathbf{m}\,\alpha + \mathbf{n}\,\beta = \mathbf{0}$ is changed to $\mathbf{n}\,\alpha + \mathbf{m}\,\beta = \mathbf{0}$, this second line reverses the angles with the two sides and is the first line reflected across the angle bisector.

These equations are homogeneous

Equations such as $\mathbf{m}\,\alpha + \mathbf{n}\,\beta = 0$ have a special property. There is no constant term; the truth of the equation is unchanged if we multiply corresponding quantities such as **m** and **n** by the same constant. This leads to a special form of equivalence between equations which seems strange at first.

Example: $\mathbf{m}\,\alpha + \mathbf{n}\,\beta = 0$ is equivalent to $\alpha/\mathbf{n} + \beta/\mathbf{m} = 0$ (divide by **mn**).

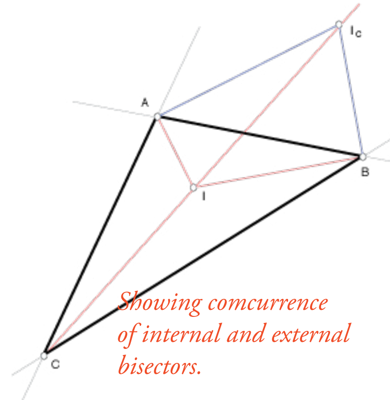
Lines related to triangles

If a third line γ is introduced (being the third side of a triangle), then any line in the plane can be written as $\mathbf{m}\,\alpha + \mathbf{n}\,\beta + \mathbf{p}\,\gamma = \mathbf{0}$. It is convenient to use the sign convention that for all three lines the positive direction will be towards the center of the triangle. With this convention, the three internal bisectors of the triangle are $\alpha - \beta, \beta - \gamma$, and $\alpha - \gamma$. The three external bisectors are $\alpha + \beta, \beta + \gamma$, and $\alpha + \gamma$.

This formalism is deep. Major proofs can now be done very simply.

Theorem: *The internal bisectors of a triangle are concurrent.*

Proof: The angle bisectors are **a - b**, **b - g**, and **g - a**. Since **g - a** = **-1**(**a -b**) **- 1**(**b - g**), the line **g - a** goes through the intersection of **a - b** and **b - g**.



Showing comcurrence of internal and external bisectors.

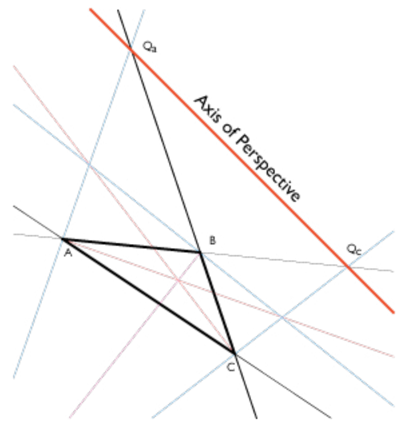
Theorem: *Two external and one internal bisector are concurrent.*

Proof: Let $\alpha+\gamma$ and $\alpha+\beta$ be the external bisectors. Then $\beta-\gamma= \mathbf{1}(\alpha + \beta) - \mathbf{1}(\alpha+\gamma)$ and the three lines $\alpha+\gamma, \alpha+\beta, \beta-\gamma$ concur.

Note: there are three versions of this, one for each internal bisector. This means that there are four natural concurrences in and around each triangle.

This is neat because it is so simple. There is more, however. Consider the line $\alpha+\beta+\gamma$. Since it can be written as $1(\alpha+\beta)+1\gamma$, this line goes through the intersection of external bisector $\alpha+\beta$ with the side γ(point $\mathbf{Q_c}$ in picture). Similarly, by using the associative property in different ways, this line goes through the two other points where external bisectors intersect the sides (points $\mathbf{Q_a}$ and $\mathbf{Q_b}$ in the picture). This proves that these three points are colinear. This line is important and is known as the *axis of perspective.*

Similarly $\alpha+\beta-\gamma$ goes through (a) $(\alpha-\gamma)+\beta$ the intersection of angle bisector $\alpha-\gamma$ with side β. (b) $\alpha+(\beta-\gamma)$, the intersection of angle bisector $\beta-\gamma$ and side α. (c) $(\alpha+\beta)-\gamma$, the intersection of external bisector $\alpha+\beta$ and side γ. Hence these three points are colinear.



Similarly for $\alpha-\beta+\gamma$ and $-\alpha+\beta+\gamma$, so that there are 4 lines each formed from 3 colinear points.

The line $\alpha+\beta+\gamma$ has a special name. It is called the *axis of perspective*.

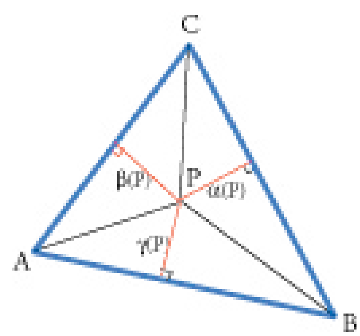
A normalization condition

There is a formula relating the three lines that determine the triangle.

For any point in the plane $\mathbf{a}\,\alpha + \mathbf{b}\,\beta + \mathbf{c}\,\gamma = \mathbf{2A}$ where **a**, **b** and **c** are the sides of the triangle and **A** is its area. This is a nice result which follows from the fact that α, β, and γ can be interpreted as distance. If the point **P**(**x, y**) is inside the triangle, lines can be drawn to the three vertices, dividing the triangle into three areas which give the above formula. In this $\alpha(\mathbf{P}), \beta(\mathbf{P})$, and $\gamma(\mathbf{P})$ are the altitudes of the triangles. If **P**(**x ,y**) is outside the triangle, the fact that some distances $\alpha(\mathbf{P})$ are negative insures the continuing correctness of the formula.

A deep result of this normalization condition is that the degree of a term in an equation can be changed at will. For example the expression $\alpha + \mathbf{k}$, which appears to be non-homogeneous, can be rewritten as $\alpha + \mathbf{k/2A}\,(\mathbf{a}\,\alpha + \mathbf{b}\,\beta + \mathbf{c}\,\gamma)$, which is homogeneous. This implies that any equation that is not homogeneous can be changed to one that is.

The normalization condition also lets us see the nature of the line $\mathbf{a}\,\alpha + \mathbf{b}\,\beta = \mathbf{0}$. We know that it goes through vertex **C**, the intersection of **a** and **b**. From the normalization condition we have $\mathbf{a}\,\alpha + \mathbf{b}\,\beta = \mathbf{2A} - \mathbf{c}\,\gamma$. The right side is parallel to side γ. Hence $\mathbf{a}\,\alpha + \mathbf{b}\,\beta = \mathbf{0}$ goes through vertex **C** parallel to side γ.



Points and lines around triangles, the overall picture

We have used the example of a triangle with its internal and external bisectors, showing concurrences and collinearities. This example is a model for understanding other common lines in the triangle such as medians and altitudes. Here too there are external medians and altitudes as well as an analogous set of concurrences and collinearities. To summarize what we discovered, each triangle has

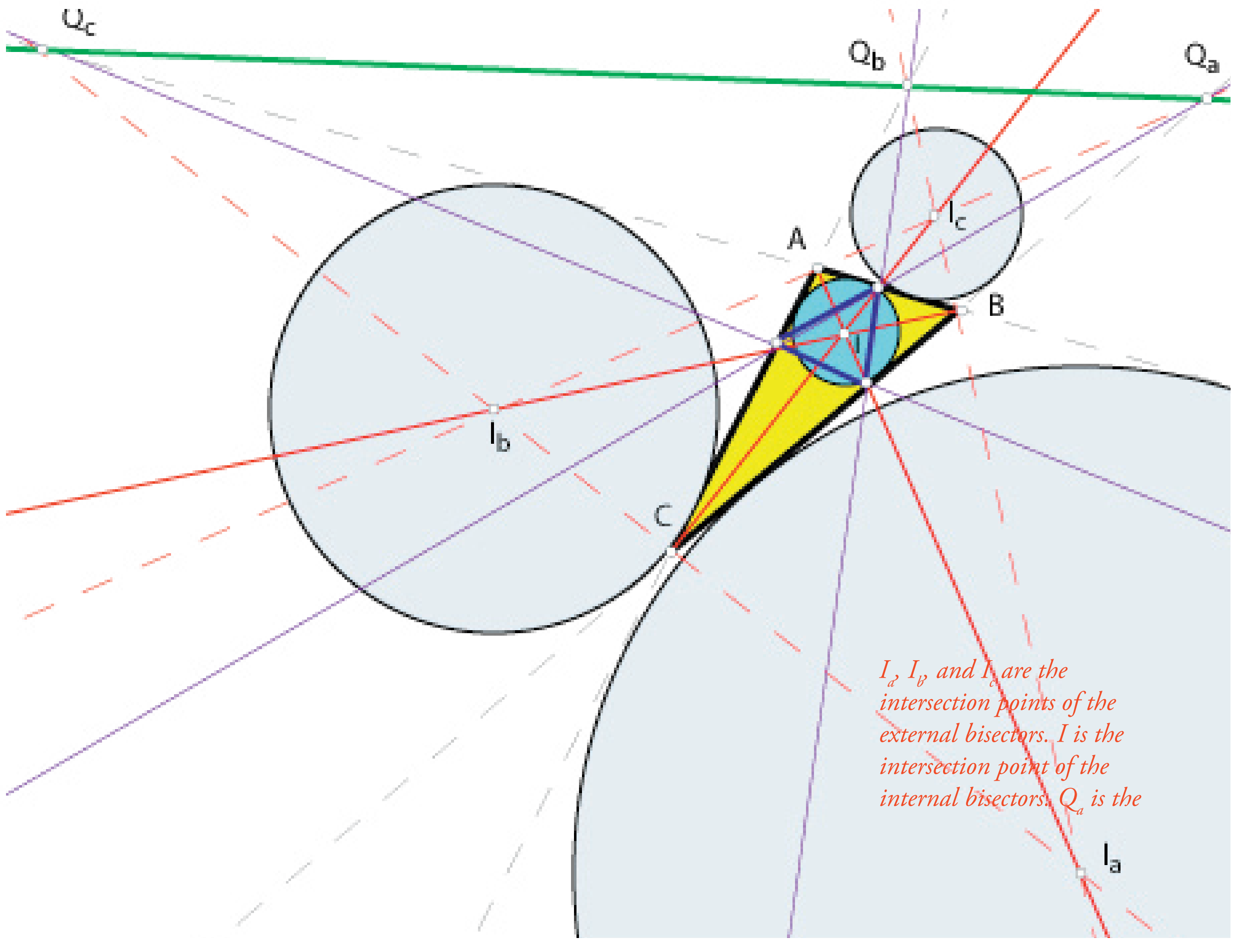
3 internal bisectors; e.g., $\alpha-\beta, \beta-\gamma, \gamma-\alpha$

3 external bisectors; e.g., $\alpha+\beta, \beta+\gamma, \gamma+\alpha$

4 points of concurrence (the three internal bisectors at **I**, and others at $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$ from the external bisectors taken two at a time, combined with one internal bisector).

4 collinearities ($\alpha \pm \beta \pm \gamma$)

Two tringles, one inscribed, one escribed.



This is such an interesting picture. There is lots more structure here. First, notice the exterior ΔIaIbIc, formed from the external bisectors. I, the incenter of the original triangle, is the orthocenter of the exterior triangle. The line AI is an internal bisector of ΔABC and an altitude of $\Delta I_aI_bI_c$. The sides of ΔABC are tangent to the three excircles. In fact each side is internally tangent to one pair of circles and externally tangent to another. These tangents/sides go through the points Q_a, Q_b, and Q_c that determine the axis of perspective. There are three other tangent lines, not drawn in the picture, each of which also goes through one of these points.

This dual structure holds just as well for **altitude**-

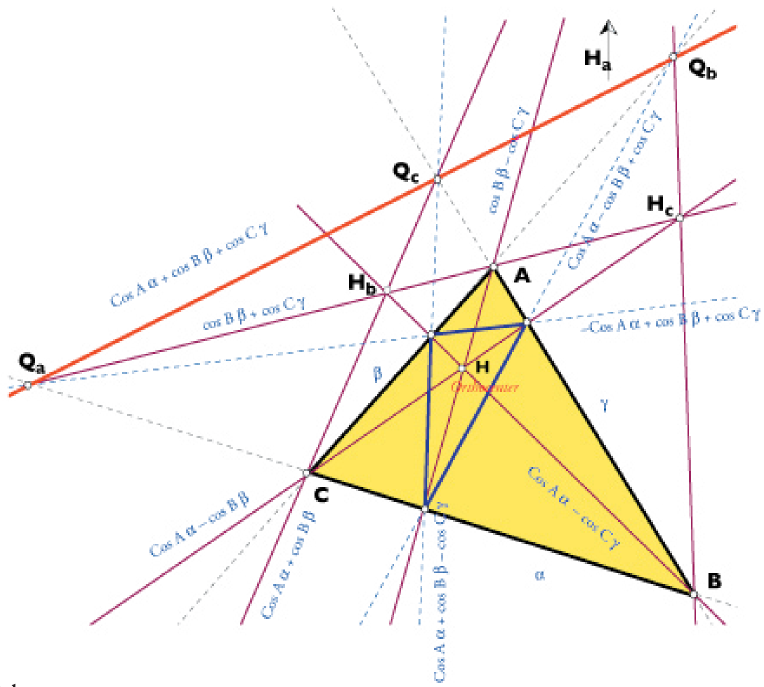
3 internal altitudes; e.g., **cos A α - cos B β, cos I**

3 external altitudes **cos A α + cos B β, cos B β +**

4 concurrences **H, H_a, H_b, H_c**

4 collinearities **cos A α ± cos B β ± cos C γ**

An inscribed and an escribed triangle ($\Delta H_a H_b H$



The inscribed triangle is known as the orthic triangle. The axis of perspective is the orthic axis. The orthic axis is perpendicular to the Euler line and is the radical axis of the circumcircle and the 9 point circle.

Also for medians

3 internal medians: **a α - b β, b β - c γ, c γ -**

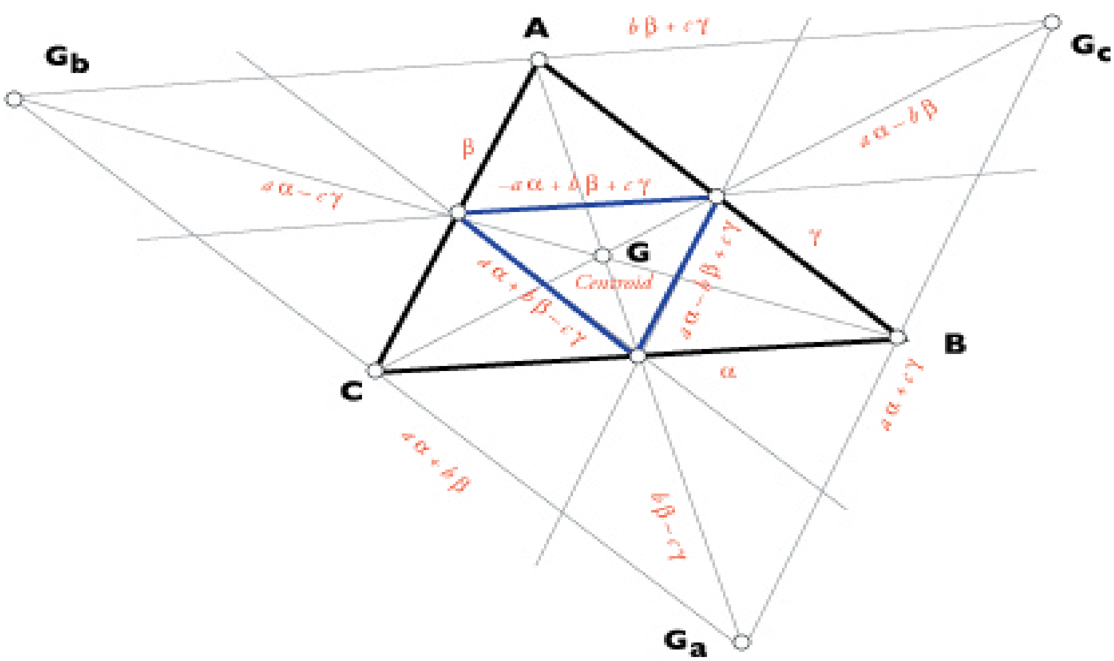
3 external medians: **a α + b β, b β + c γ, c γ**

(we have already seen that these are the lines thr vertices parallel to opposite sides of the triangle)

4 points of concurrence including the centroid.

4 collinearities (one, the axis of perspective, is the line at infinity): **a α ± b β ± c γ**

An inscribed triangle (the medial triangle) and an escribed triangle ($\Delta G_a G_b G_c$)



Notice all the parallel lines in this situation. There is no axis of perspective (or equivalently, the axis is the line at infinity) because the requisite lines do not meet. The inscribed triangle is called the medial triangle which is particularly important in the geometry of the triangle. The circumcircle of the medial triangle is the 9 point circle. The incircle of the medial triangle is the Spieker circle.

Lines related to triangles

Here are the equations of some of the more famous lines in a triangle. They are listed both to give a flavor for the sorts of equations one is likely to deal with and to show that many of these equations have a particularly simple form. Our distance convention is that for each side of the triangle the distance towards the center of the triangle is positive.

Median through **C**: $\mathbf{Sin\ A \cdot \alpha - Sin\ B \cdot \beta = 0}$ or $\mathbf{a\,\alpha - b\,\beta = 0}$

Symmedian through **C**: $\mathbf{b\,\alpha - a\,\beta = 0}$ **or** $\mathbf{\alpha / a - \beta / b = 0}$

Altitude through **C**: $\mathbf{Cos\ A \cdot \alpha - Cos\ B \cdot \beta = 0}$

Perpendicular bisector through side **c**: $\mathbf{Cos\ A \cdot \alpha - Cos\ B \cdot \beta - c/2\ Sin(B - A) = 0}$

Joins feet of altitudes on **AC** and **BC**: $\mathbf{Cos\ A \cdot \alpha + Cos\ B \cdot \beta - Cos\ C \cdot \gamma = 0}$

Line through midpoints of **AC** and **BC**: $\mathbf{a\,\alpha + b\,\beta - c\,\gamma = 0}$. Note that since $\mathbf{a\,\alpha + b\,\beta + c\,\gamma = 2A}$ we can write this equation as $\mathbf{c\,\gamma = A}$ so that this line is the parallel to γ through C.

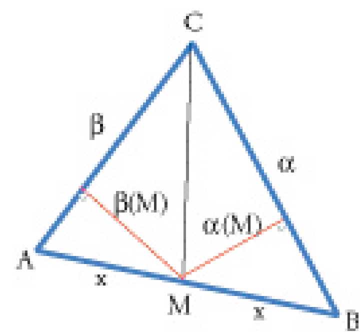
Line that joins the feet of the altitudes from **A** and **B**: $\mathbf{Cos\ A \cdot \alpha + Cos\ B \cdot \beta - Cos\ C \cdot \gamma = 0}$

Euler line: $\mathbf{Sin\ 2A\ Sin(B - C) \cdot \alpha + Sin\ 2B\ Sin(C - A) \cdot \beta + Sin\ 2C\ Sin(A - B) \cdot \gamma = 0}$

Joins incenter and circumcenter: $\mathbf{(Cos\ B - Cos\ C) \cdot \alpha + (Cos\ C - Cos\ A) \cdot \beta + (Cos\ A - Cos\ B) \cdot \gamma = 0}$

We can verify the equation of the median through **C** as follows. The median through **C** (the intersection of α and β) can be written as $\mathbf{m}\,\alpha - \mathbf{n}\,\beta = 0$. Evaluated at M (see figure) we have $\mathbf{m}\,\alpha(M) - \mathbf{n}\,\beta(M) = 0$. From the picture we can see that $\alpha(M) = \mathbf{x\ Sin\ B}$ and $\beta(M) = \mathbf{x\ Sin\ A}$. From this we get that, up to a proportionality, $\mathbf{m = Sin\ A}$ and $\mathbf{n = Sin\ B}$. hence the median through C is $\mathbf{Sin\ A \cdot \alpha - Sin\ B \cdot \beta = 0}$. Note: since these equations have no constant terms, most variables can only be computed up to a proportionality.

It is not always easy to determine if two equations are equivalent. For example we gave two forms for medians: $\mathbf{Sin\ A \cdot \alpha - Sin\ B \cdot \beta = 0}$ or $\mathbf{a\,\alpha - b\,\beta = 0}$. These are equivalent because the first equation can be multiplied by **2R**, where **R** is the radius of the circumcircle. Then we use the law of sines $\mathbf{2R\ sin\ A = a}$ to change from one equation to the other.



General Concurrences

The form of these equations guarantees that all three of similarly defined lines (such as bisectors, altitudes, and medians) will meet at a common point.

Theorem: *If three lines are defined as* $\mathbf{m}\,\alpha - \mathbf{n}\,\beta = 0$, $\mathbf{n}\,\beta - \mathbf{p}\,\gamma = 0$, *and* $\mathbf{p}\,\gamma - \mathbf{m}\,\alpha = 0$, *where **m,n,** and **p** are functions of the sides and angles of the reference triangle, the three lines are concurrent.*

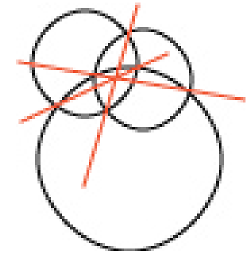
Proof: **line#1** = **–1 · line#2 – 1· line#3**, hence **line#1** goes thought the intersection of **line#2** and **line#3.** QED. Concurrencies and colinearities are built into this notation for lines. Note that it only took one line to simultaneously prove that the medians and the altitudes and the symmedians are concurrent. This theorem works just as well if we substitute functions of sides for functions of angles. It can also be generalized to functions of the form **f(a, b, c)** if they are symmetric in **a** and **b**.

The isogonal concurrency

We learned before that if $\mathbf{m\,\alpha - n\,\beta = 0}$ is changed to $\mathbf{n\,\alpha - m\,\beta = 0}$, this second line reverses the angles with the two sides and is the first line reflected across its angle bisector (through **C** in this case). If we do this for three lines concurrent at **P**, the three reflected lines will be concurrent at a new point **P'**. This point is called the isogonal conjugate of **P**. Among points we know the othocenter and the circumcenter are isogonal conjugates.

Another famous conjugate is the symmedian point, which is the isogonal conjugate to the centroid.

Bonus: the radical axes of three circles are concurrent



A circle can be written in the form $\mathbf{x^2 + y^2 + a\,x + b\,y + c = 0}$. Let C_1, C_2, and C_3 be circles in this form. Then $\mathbf{C_1 - C_2 = 0}$ is a straight line because the quadratic terms cancel. If the circles intersect, this line goes through the points of intersection of the two circles. In any case this line is called the radical axis of the two circles. For three circles taken pairwise there are three radical axes.

Theorem: *The radical axes of three circles are concurrent.*

Proof: $\mathbf{C_1 - C_2 = 1(C_1 - C_3) + 1(C_3 - C_2)}$, hence the line $\mathbf{C_1 – C_2}$ goes through theintersection of the other two.

Some Advanced Geometry

As we read books that take us past the familiar geometry learned in high school, we come across confusing terms, often named after their discoverers: the Lemoine line, the Nagel point, the Euler line, the Gergonne point, the Nagel line, the tritangent axis, the symmedian point. We now possess the conceptual apparatus to make some sense of these terms. The above named points are all concurrences of three lines drawn from the vertices of the triangle. Our formalism shows that concurrences should be common. In fact if the definition of a geometric object is sufficiently symmetric to the three sides of a triangle, there should be a concurrence.

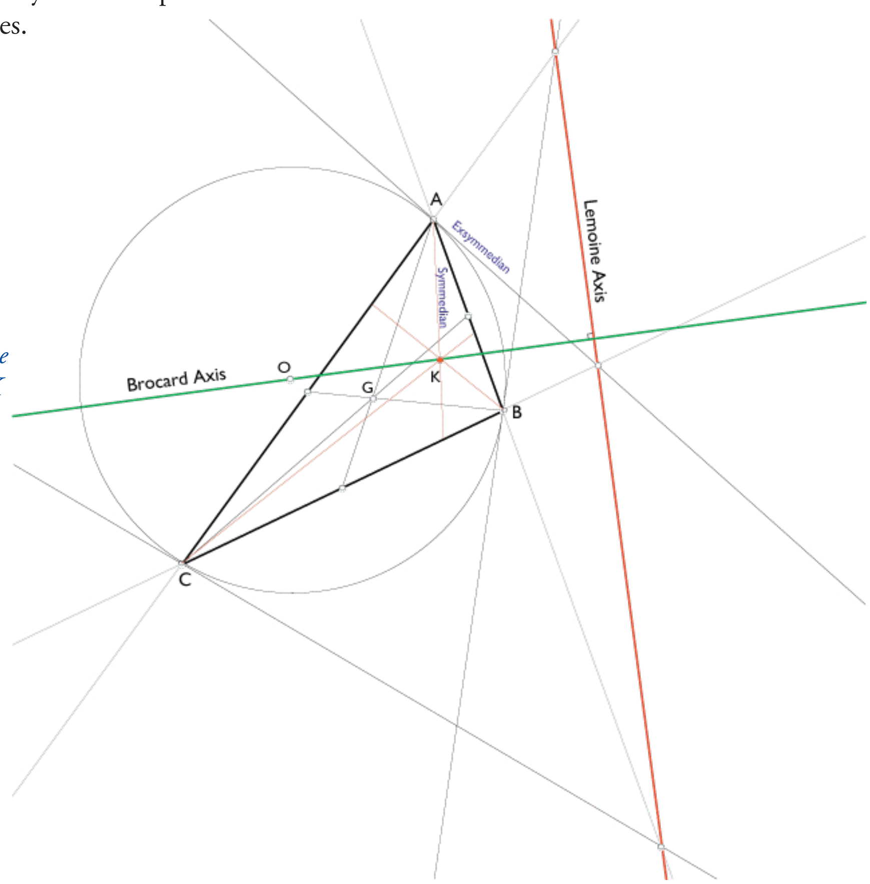
The Gergonne point is the concurrence of the cevians to the contact points made by the incircle with the triangle. The Nagel point is the concurrence defined by the contact points of the three escribed circles.

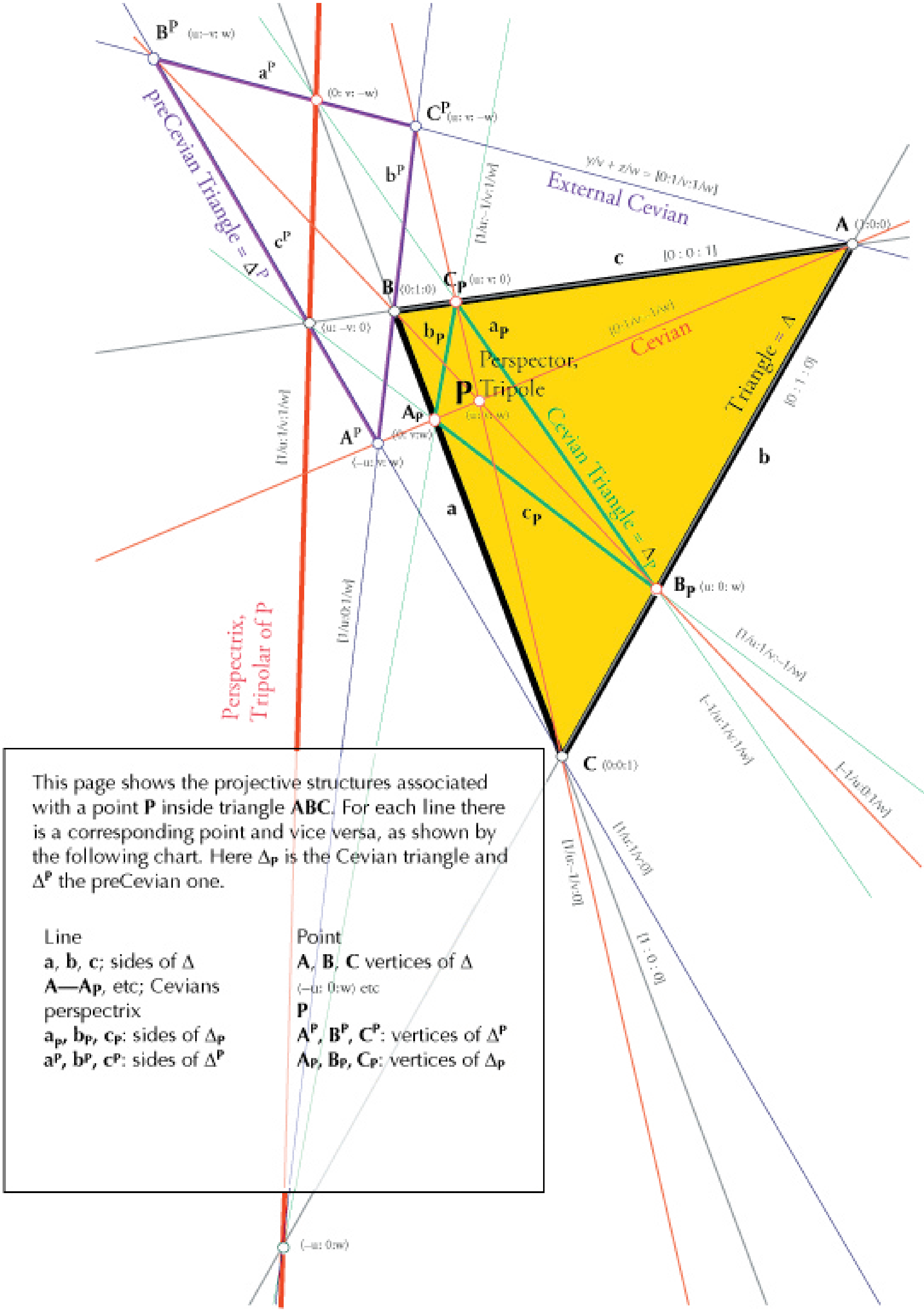
The symmedian point is the isogonal conjugate of the centroid. Other than the incenter, all points based on concurrencies will have yet another concurrency, its isogonal conjugate.

We learned that all concurrencies inside the triangle have a colinearity outside the triangle called the axis of perspective. Two of the lines named above, the Lemoine line and the orthic axis are axes of perspective for some concurrency inside the triangle.

The Lemoine line is the axis of perspective of the symmedian point. The orthic axis is the axis associated with the altitudes.

This picture shows some of what is called the Lemoine Geometry of the triangle. The isogonal conjugate of the centroid G is the symmedian point K. Cevians going through this point are called symmedians and are antiparallel to the opposite side. The corresponding exsymmedians are the tangents to the circumcircle at the vertices. The Lemoine axis is the axis of perspective of K and is perpendicular to the line between O and K.





This page shows the projective structures associated with a point **P** inside triangle **ABC**. For each line there is a corresponding point and vice versa, as shown by the following chart. Here Δ_P is the Cevian triangle and Δ^P the preCevian one.

Line	Point
a, **b**, **c**; sides of Δ	**A**, **B**, **C** vertices of Δ
A—A_P, etc; Cevians	(–u: 0:w) etc
perspectrix	**P**
a_P, b_P, c_P: sides of Δ_P	A^P, B^P, C^P: vertices of Δ^P
a^P, b^P, c^P: sides of Δ^P	A_P, B_P, C_P: vertices of Δ_P