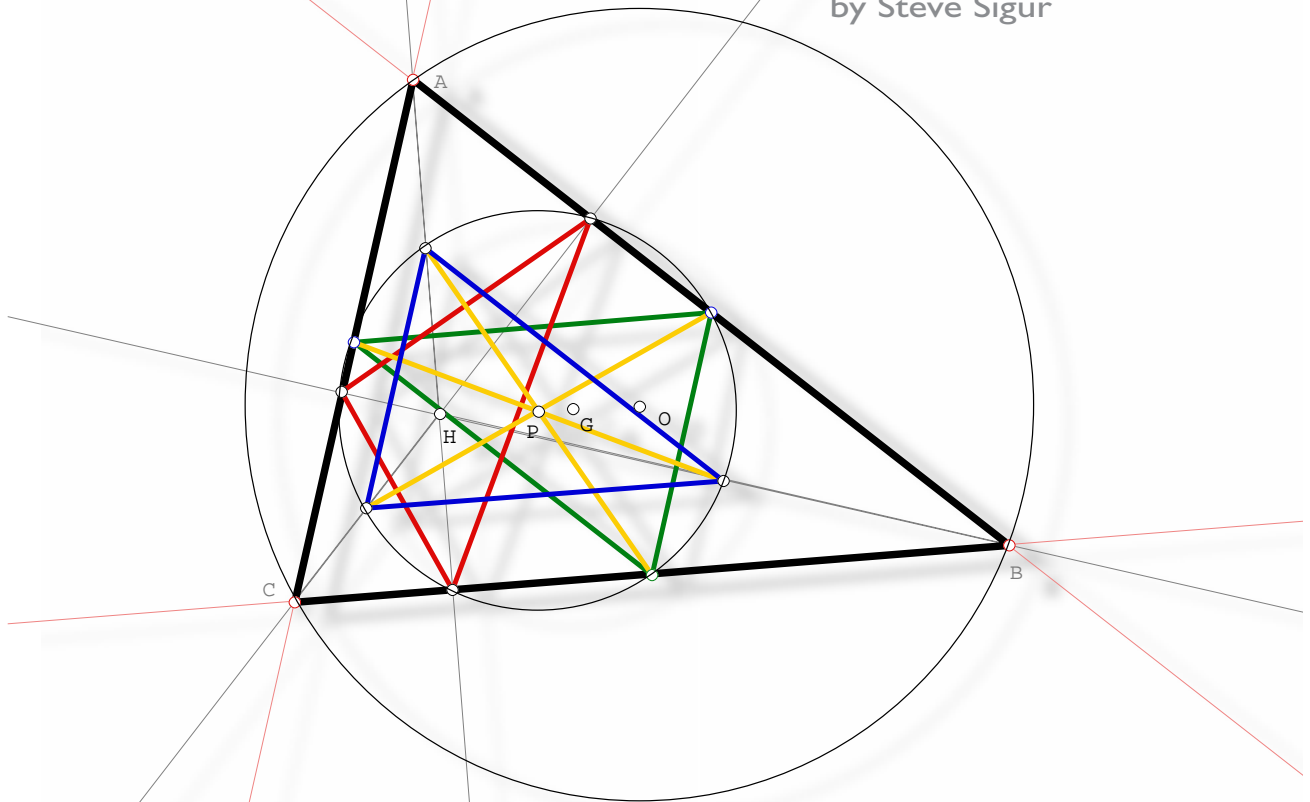


The Modern* Geometry of the Triangle

by Steve Sigur



This essay is designed to get you up to speed so you can read John Conway's descriptions of triangle geometry. These pages are designed to be viewed side by side, text on the left, pictures on the right. Acrobat can be set to view pages in this way.

© 1999 Steve Sigur. Permission given for non-profit use.

*In this context "modern" means less than 200 years old

In school we learn a geometry of equalities, based on quantities being exactly equal in some way. But many of the structures in geometry are not best expressed distances, angles, or congruencies. In this little essay, I try to present a more modern view of triangle geometry based on the appreciation of structures in and about triangles. To do this we start with the most fundamental properties of points and lines—properties that arise from nothing other than “pointness” and “lineness” — and proceed to “turn on” other aspects a few at a time. This is not a formal exposition and everything here is presented without proof. Nor is it thorough. I only want to give those interested in geometry the perceptual and vocabulary background to be able to follow Conway’s posting about geometry. I suggest that you play with each topic; draw the pictures, find relationships, prove theorems.

Steve

The dual nature of points and lines

We begin by going to the core of geometry. We are going to “turn off” most properties we usually use and keep only two. Two distinct points determine a line. Two distinct lines determine a point. If the lines are parallel, the point is at infinity. Included in what we turn off are all indications of middle-ness such as midpoint, angle bisector, and perpendicular bisector. Likewise we will measure no distances or angles. As we inhabit what seems to be a very sparse geometrical world, we will be surprised at all the structure we find, structure that was there all along but that we usually miss.

Begin by putting a point in the plane. Adding a second point allows the creation of a line. Adding a third point allows the creation of two more lines. Now we have three points and three lines but no more. When we add the 4th point, all hell breaks loose. We get structure after structure.

Let’s show this structure a step at time. It will be helpful for you to draw the structures as we describe them. Remember that all lines extend to infinity.

We start with 3 lines and 3 points. Add a 4th point inside the triangle. Where the point is placed does not matter but it will help your visualization if the point is inside the triangle formed by the three lines.

Call the point P and the vertices of the triangle A , B , and C . With the addition of P three more lines AP , BP , and CP are created. They are called cevian lines. These three lines intersect the sides of triangle ABC in three new points A_p , B_p , C_p . These three points form a new triangle, the cevian triangle. Now this is all very nice but the surprises are yet to come.

The sides of this new triangle $A_pB_pC_p$ intersect the corresponding sides of ABC in three new points Q_a , Q_b , Q_c . The first surprise is that Q_a , Q_b , and Q_c never form a triangle; these points are colinear, an indication of new structure. The points $Q_aB_pC_p$, $Q_bA_pC_p$, and $Q_cA_pB_p$ are also colinear, making Q_a , Q_b , and Q_c concurrences.

The line through Q_a , Q_b , and Q_c is called the axis of perspective or the tripolar line (one thing about geometry is that there is a lot of confusing terminology).

Now draw the lines A^pA , B^pB , C^pC . These lines are called the external cevians or excevians. They go through the vertices of ABC and so form a new triangle (the precevian triangle) circumscribed about ABC , the precevian triangle. This generates three new concurrences, e.g., A_pA , B^pB , and C^pC concur. Each of the other two cevian lines leads to a similar concurrence. Note: this notation using P as a subscript for vertices of the cevian triangle and as a superscript for the precevian triangle was introduced by John Conway.

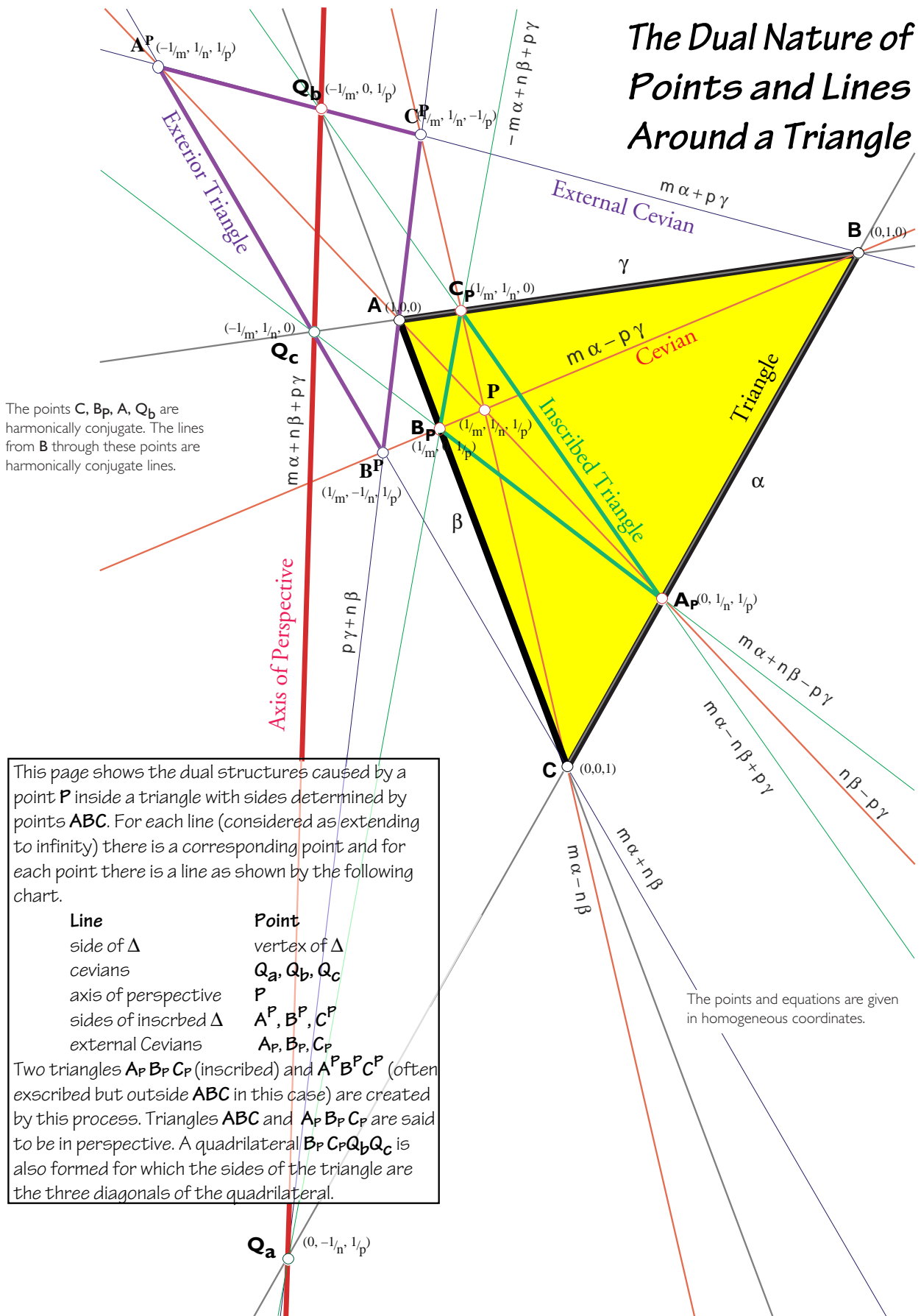
This net of points and lines grows forever. This structure happens for every point in the plane of the triangle. Notice that we have only used lines and points in the plane and the dual rules allowing the creation of new points and lines. Quite a structure from such humble beginnings.

Triangles in perspective

If the vertices of two triangles can be paired and connected so that the connecting lines are concurrent. We say that the triangles are in perspective. The point of concurrency is called the perspector. The axis of perspective is often called the perspectrix. For triangles in perspective all the above structure exists.

Triangles are commonly in perspective. This is a major structural element in geometry.

The Dual Nature of Points and Lines Around a Triangle



The points C, B_P, A, Q_B are harmonically conjugate. The lines from B through these points are harmonically conjugate lines.

This page shows the dual structures caused by a point P inside a triangle with sides determined by points ABC . For each line (considered as extending to infinity) there is a corresponding point and for each point there is a line as shown by the following chart.

Line	Point
side of Δ	vertex of Δ
cevians	Q_a, Q_b, Q_c
axis of perspective	P
sides of inscribed Δ	A^P, B^P, C^P
external Cevians	A_P, B_P, C_P

Two triangles $A_P B_P C_P$ (inscribed) and $A^P B^P C^P$ (often exscribed but outside ABC in this case) are created by this process. Triangles ABC and $A_P B_P C_P$ are said to be in perspective. A quadrilateral $B_P C_P Q_B Q_C$ is also formed for which the sides of the triangle are the three diagonals of the quadrilateral.

The points and equations are given in homogeneous coordinates.

Affine geometry

Senses of center are important so we now add our first sense of middleness, the midpoint between two points. Affine properties of triangles are independent of the shape of the triangle or the lengths of the sides, hence distances and angles are not considered.

Affine transformations change the shape of the triangle while preserving the distribution of points (as ratios) on any given line. Properties that do not change with these transformations, and hence are independent of the shape of the triangle, are affine properties. Concurrency and colinearity are not changed. Parallel lines remain parallel.

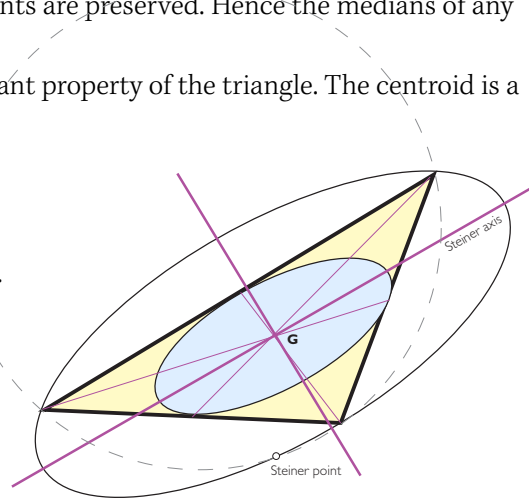
Here is an affine proof that the medians of a triangle are concurrent. Consider an equilateral triangle with the three medians drawn. By symmetry they concur in the center of the triangle. Now transform the triangle and the lines into any other triangle. Since ratios are preserved, midpoints are preserved. Hence the medians of any triangle concur.

The centroid, dividing the medians in a 2:1 ratio, is an affine invariant property of the triangle. The centroid is a very important point.

The Steiner ellipses

To an equilateral triangle add its circumcircle and its incircle. Transform this into any other triangle. The circles become ellipses. Hence any circle has an inscribed ellipse that touches the triangle at its midpoints. The two ellipses share the same axes, called the Steiner axes of the triangle.

The Steiner point is the fourth point of intersection of the circumscribed Steiner ellipse and the circumcircle.



Harmonic conjugates

Point B is between A and C, dividing AC in the ratio AB/BC . There is another point for which these ratios are the same.

Points in a line $A-B-C-D$ are said to be harmonic conjugates if $AB/BC = -AD/BD$.

4 lines emanating from the same point are harmonic conjugates if the 4 points determined by a transversal are harmonically conjugate.

Using the notation from the first two paragraphs of this essay, $A C_p B C^p$ are harmonically conjugate points.

CA, CC_p (the cevian), CB, CC^p (the excevian) are harmonically conjugate lines. The excevians are sometimes called the harmonic conjugate lines.

Special points and lines

A point in the plane of the triangle becomes special if unexpected structure occurs in association with it. Such structure is often indicated by three lines being concurrent at a point. Since 3 lines drawn randomly have zero probability of being concurrent, concurrence (and colinearity) are taken as evidence of interesting structure.

The classical centers are all defined by the concurrence of important lines in the triangle. These centers and the cevians that create them are

<i>Centroid</i>	<i>medians</i>
<i>Incenter</i>	<i>angle bisectors</i>
<i>Circumcenter</i>	<i>perpendicular bisectors</i>
<i>Orthocenter</i>	<i>altitudes</i>

Others discovered more recently (about 100 to 150 years ago) are

Associated with the incenter

Gergonne point.

Nagel Point.

Spieker center

cevians to contact points of incircle

cevians to contact points of excircles

incircle to medial triangle.

Associated with the centroid

Steiner point

Tarry point

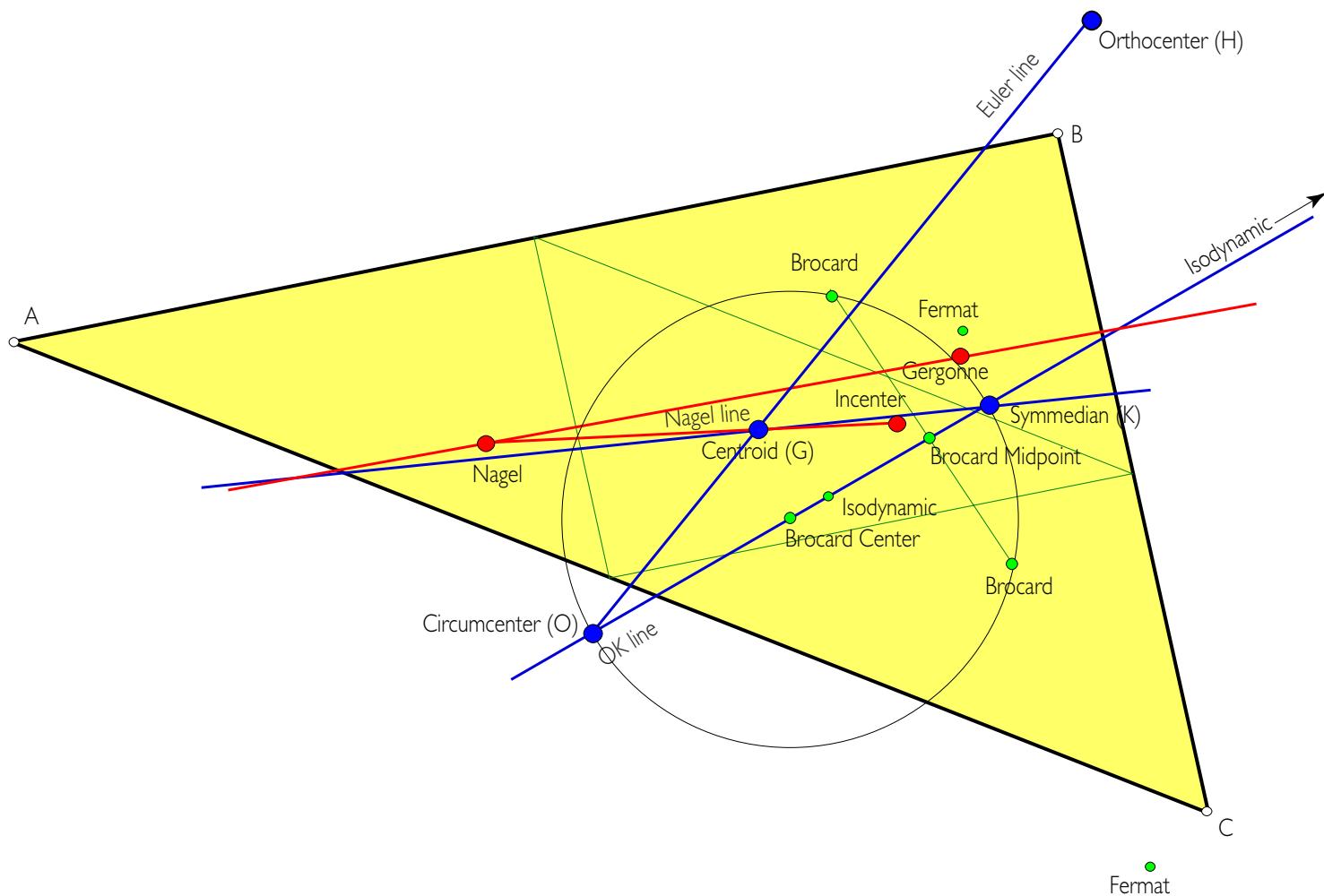
9 pt center

explained in section on Brocard Δ

explained in section on Brocard Δ

center of 9 pt circle

Points that occur in pairs such as the isodynamic and isogonal points.



Around ΔABC are more triangles

Cevians in triangles lead to the previously described structure of points and lines. Remember that this structure includes an interior triangle whose vertices are the feet of the cevians and an exterior triangle whose sides are the excevians.

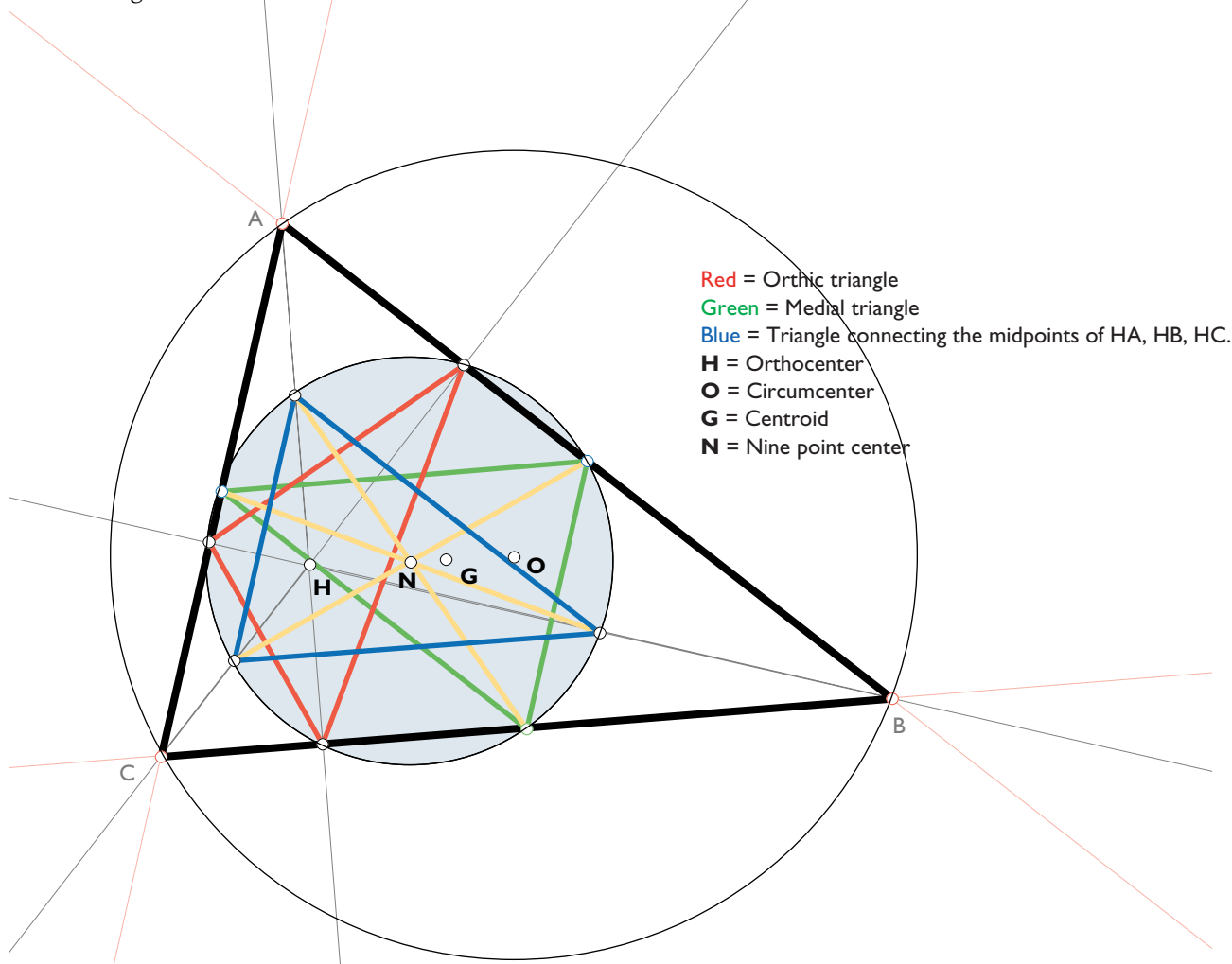
Each of the additional triangles has its own set of special points which are often special points of the original triangle.

The triangle connecting the midpoints of each side is the medial triangle. It is in perspective with ΔABC . The centroid is the perspector and the line at infinity the perspectrix. The exterior triangle is often called the antimedial triangle. The circumcenter of the medial triangle is the nine point center of the original triangle. The circumcenter of the antimedial triangle is the orthocenter of ΔABC . We will make much use of this relationship on the subsequent page "The Euler line is a piece of cake."

The triangle connecting the feet of the altitudes is the orthic triangle. It is in perspective with ΔABC . The orthocenter is the perspector and the orthic axis is the perspectrix. The incenter of the orthic triangle is the orthocenter of ΔABC .

The triangle connecting the points of contact of the incircle with the sides is the contact triangle. This triangle is in perspective with ΔABC with the Gergonne point as its perspector. The incenter of the contact triangle is the Spieker point of ΔABC .

Given a point P the triangle connecting its feet on the sides of the triangle is the pedal triangle. The pedal triangle is not generally in perspective with ABC . The pedal triangle for H is the orthic triangle. For O , the medial triangle.



Relationships between triangles

The last section mentioned the importance of triangles associated with the reference triangle. The first section pointed out that with each point P around ΔABC , there is an inscribed triangle (the cevian triangle) and a circumscribed triangle (the precevian triangle).

Triangles with the same sidelengths (Congruence)

A main topic of high school geometry, congruent triangles are less important for understanding the structural relations in triangles. Important properties of triangles tend to be scale independent.

Triangles with the same angles (Similarity)

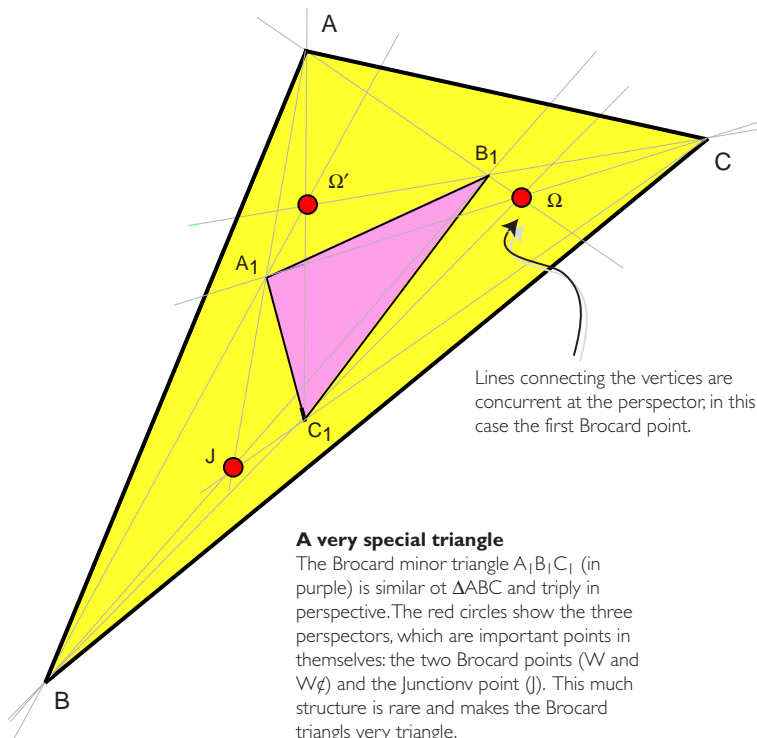
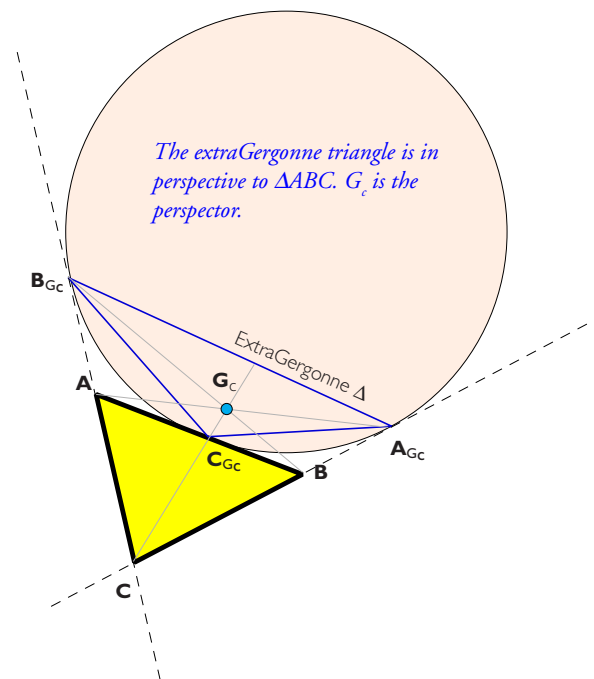
Very important. Similarity relates properties that are independent of scale or size. There are several particular types of similarity.

Homothetic triangles: a special type of similarity in which corresponding sides are parallel. For this circumstance there is a center of similarity. The medial triangle of ABC is homothetic to ABC with the centroid as the center of similarity.

Pivot point triangles (see picture on next page): Given a point P and its pedal triangle, rotate the vertices of the pedal triangle by a given angle. The resulting triangle is similar to the pedal triangle.

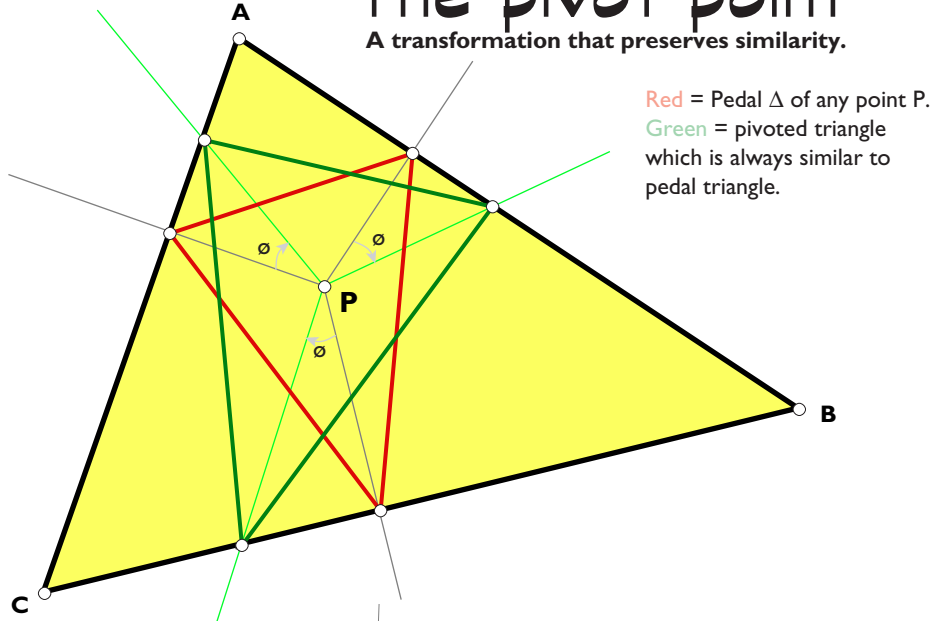
Perspective triangles

Probably the deepest relationship. If corresponding vertices are connected, the three lines are concurrent. For this relationship, all the structure of section 1 holds. The point of concurrence is called the perspector. It's corresponding axis of perspective is called the perspectrix. The cevian and precevian triangles are always in perspective with ABC .



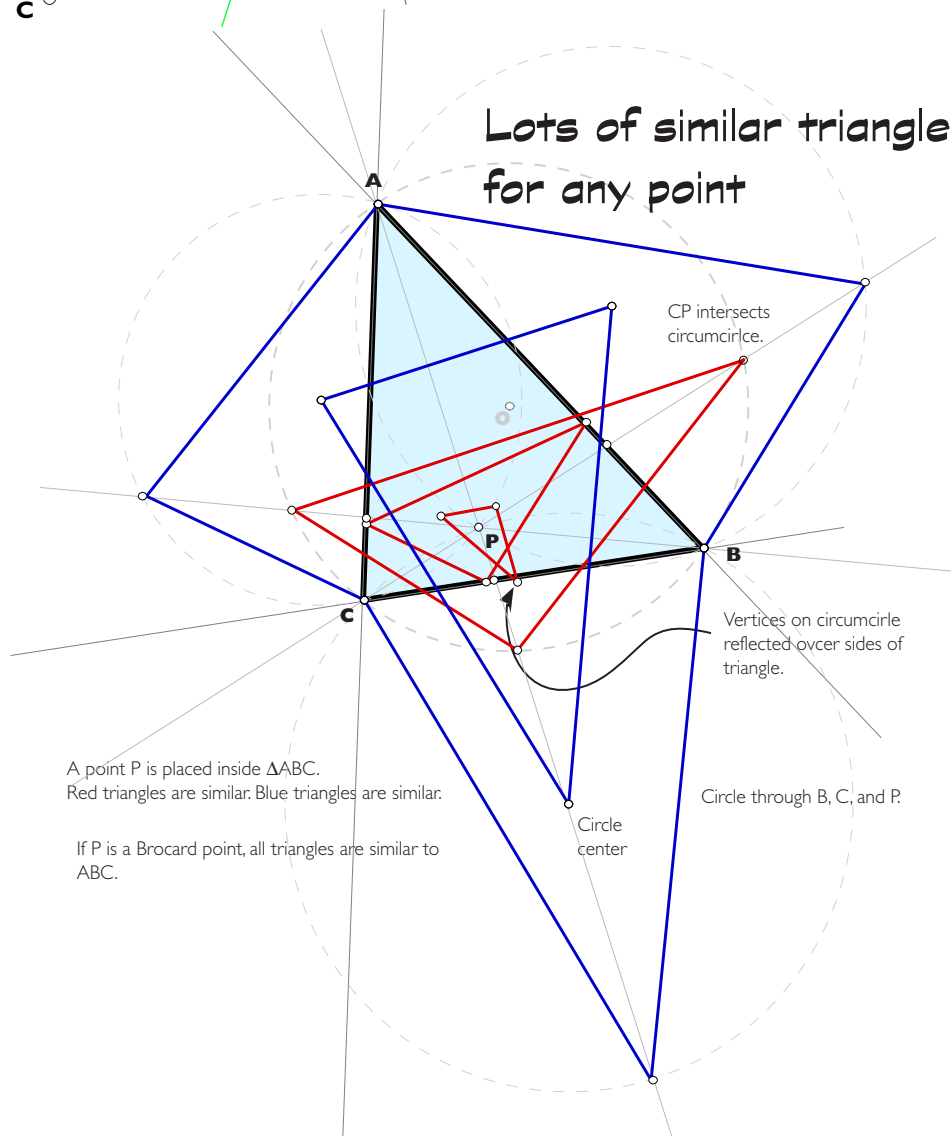
The pivot point

A transformation that preserves similarity.



Red = Pedal Δ of any point P.
 Green = pivoted triangle which is always similar to pedal triangle.

Lots of similar triangles for any point



CP intersects circumcircle.

Vertices on circumcircle reflected over sides of triangle.

A point P is placed inside ΔABC .
 Red triangles are similar. Blue triangles are similar.

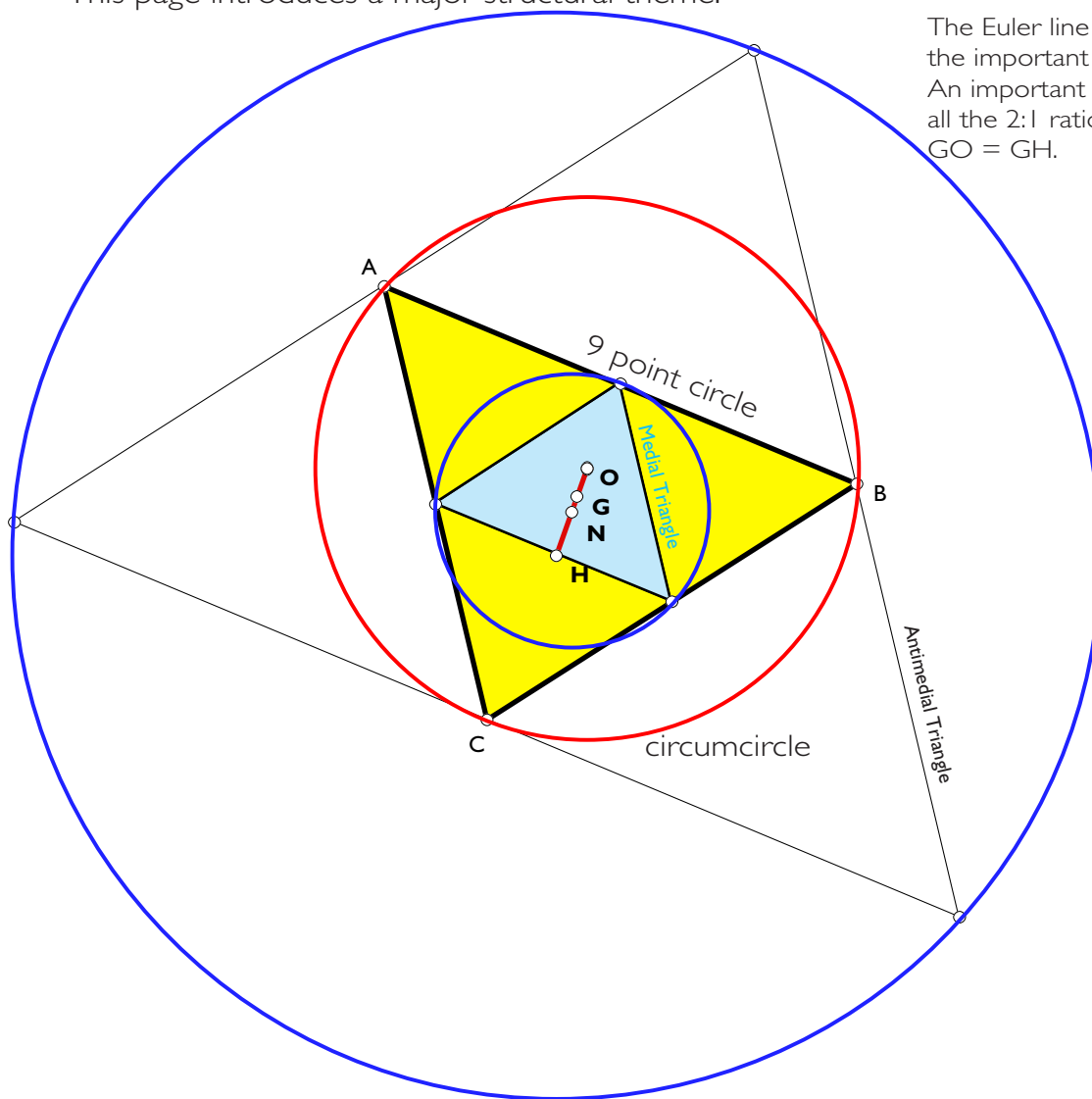
If P is a Brocard point, all triangles are similar to ABC.

Circle center

Circle through B, C, and P.

The Euler Line is a Piece of Cake

Structure given by the centroid and its medial and antimedial triangles.
This page introduces a major structural theme.

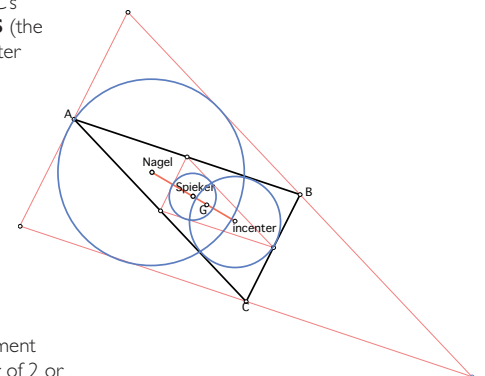


The Euler line organizes some of the important triangle centers. An important characteristic are all the 2:1 ratios. $2 NG = GO$; $2 GO = GH$.

The four points of the **Euler line** are the centroid **G** and three circumcenters: **O**, that of ABC; **P** (the 9-pt center; which is the circumcenter of ABC's medial triangle); and **H** (ABC's orthocenter; which is also the circumcenter of ABC's "antimedial" triangle). The four points of the **Nageline** are the centroid **G** and three incenters: **I**, that of ABC; **S** (the Spieker center; which is the incircle of ABC's medial triangle); and **N** (the Nagel point, which is also the incenter of ABC's "antimedial" triangle). In a chart:

Euler Line (circumcenters)	Nagel Line (incenters)
G (centroid)	G (centroid)
N (medial's circumcenter)	So (medial's incenter)
O (ABC's circumcenter)	Io (ABC's incenter)
H (antimedial's circumcenter)	No (antimedial's incenter)

"Now think of how ABC, its medial, and its antimedial triangles are related. To get the medial, rotate ABC by 180 about the centroid **G**, and then shrink all distances from **G** by a factor of 2. To get the antimedial, do the same rotation but enlarge all distances from **G** by a factor of 2. Well, when you think about what happens to **O** – ABC's circumcenter – under this transformation, you see that it turns into **P** in one case, and **H** in the other. In other words, the Euler line is simply the statement that if you take a line segment – **GO**, in this case – and rotate it by 180 about one of its endpoints (**G**), and then either shrink it by a factor of 2 or enlarge it by a factor of 2, then resulting endpoints (**P** and **H**) will be collinear with the original segment... and will have fixed distance ratios from the original endpoint...." [this last paragraph is from a posting by Douglas Hofstadter to the geometry.research newsgroup titled "The Euler Line is a piece of cake."].



The Nagel Line comes from the centers of the inscribed circles.

The isogonal and isotomic conjugates

After using concepts of centeredness, we need ideas that express equal separation from that center. There two very important ideas. One, the isogonic conjugate, defines separation via angles. The other, the isotomic conjugate, defines separation in terms of distances along the side of a triangle.

The conjugal or isogonic conjugate

Let a line from the vertex is drawn in the interior of the angle. The reflection of this line across the bisector from the same vertex is the isogonal conjugate line. In this situation a sense of center is defined (the angle bisector) and a sense of equal separation from the center (the two conjugate lines).

There is lots of structure here. To see this choose points P and Q , each on one of the conjugate lines in angle ABC (see picture on next page). Mark the feet of P, Q on the sides of the angle. A circle, centered at the midpoint of PQ , goes through all 4 points.

Now choose an analogous situation for a triangle and its three angles. Draw all three angle bisectors. Choose a point P and draw its three cevians. Now reflect those cevians, each across the bisector originating from the same vertex. The reflected cevians will be concurrent at a point Q . Q is called the isogonal conjugate of P .

There will be a circle going through the 6 points that are the feet of P and Q on the sides of the triangle. This circle will be the circumcircle of the pedal triangles of both P and Q .

Lines from the vertices perpendicular to the pedal triangle of P are concurrent at the isogonal conjugate of P .

If an important point is produced by a concurrency, its isogonal conjugate will also be an important point.

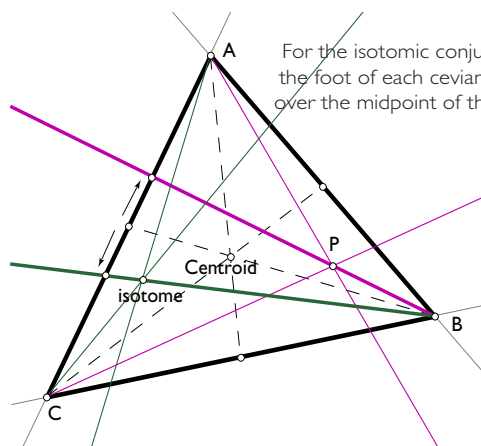
The isogonal conjugate of the circumcenter is the orthocenter. The isogonal conjugate of the incenter is itself. The isogonal conjugate of the centroid is the symmedian point. The conjugate lines to the medians are called symmedians.

The circumcircle is the isogonic conjugate of the line at infinity.

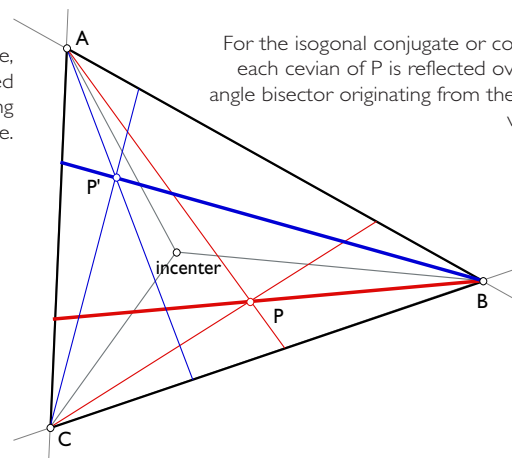
The isotome or isotomic conjugate

Consider triangle ABC with three cevians concurrent at P . Reflect the foot of each cevian across the midpoint of its side. Draw cevians to these points. The new cevians will be concurrent at P' , the isotomic conjugate of P . The most famous isotomic conjugates are the Gergonne and Nagel points.

The circumscribed Steiner ellipse is the isotomic conjugate of the line at infinity.

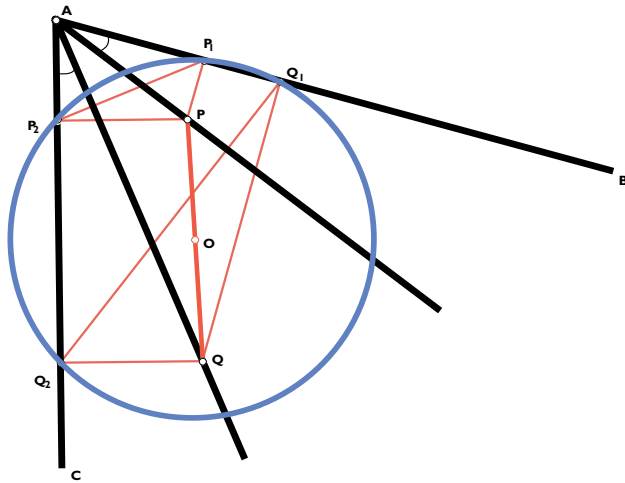


For the isotomic conjugate or isotome, the foot of each cevian of P is reflected over the midpoint of the corresponding side.



For the isogonal conjugate or conjugal, each cevian of P is reflected over the angle bisector originating from the same vertex.

Structures related to isogonic conjugacy

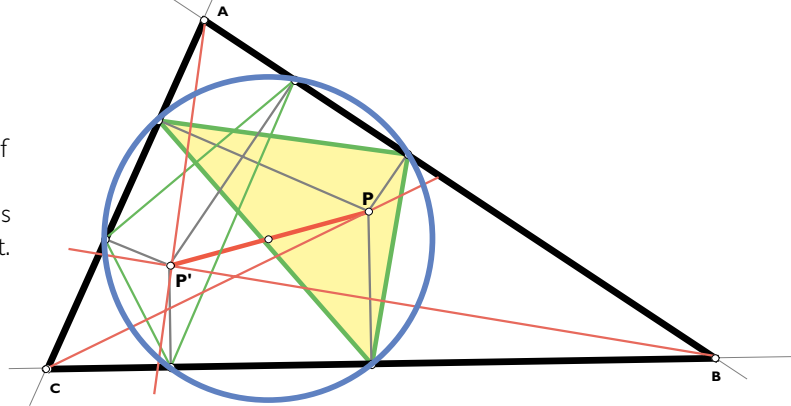


Given P and Q, points on two isogonally conjugate

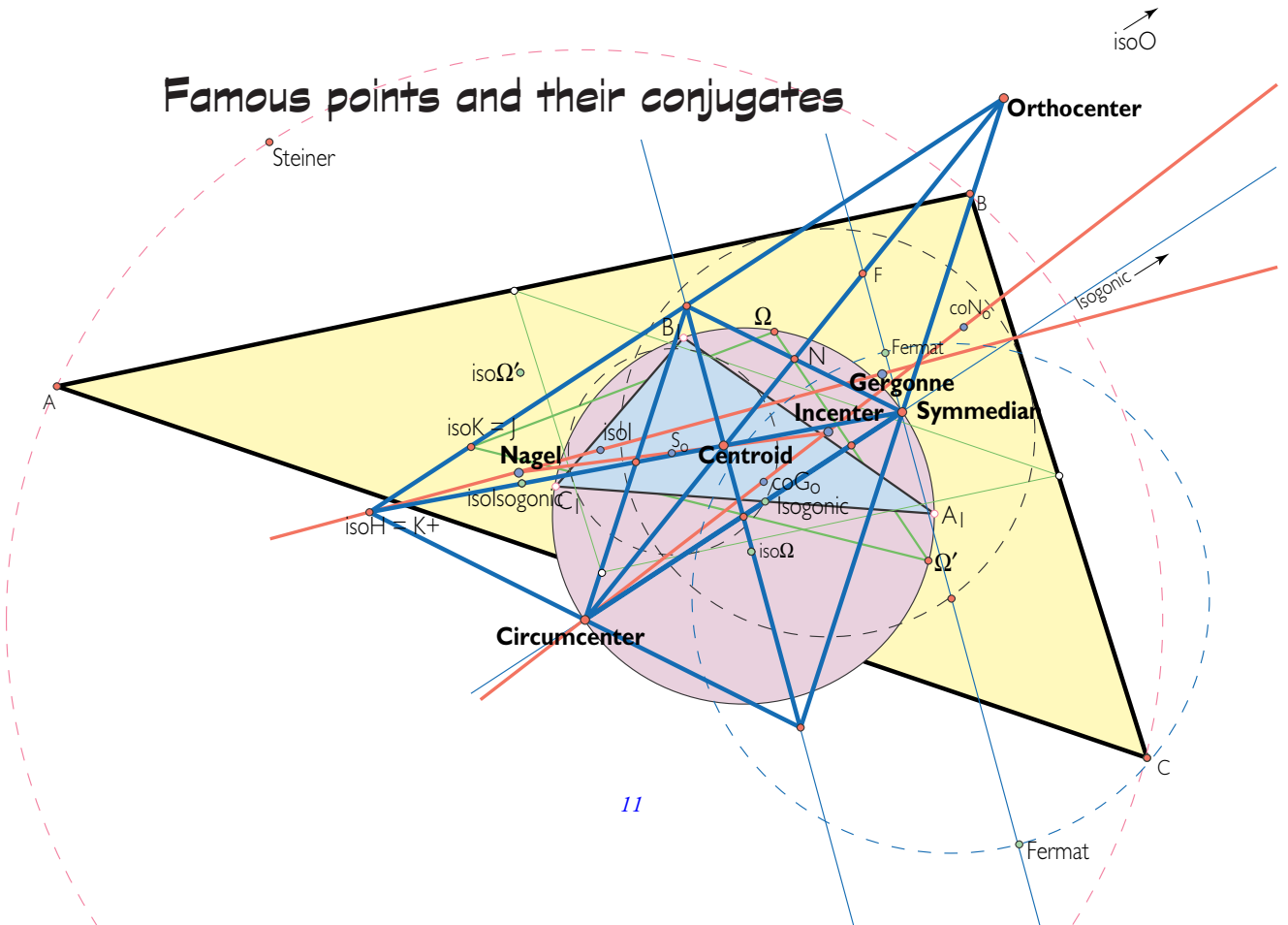
lines in $\angle ABC$. The other points are projections of P and Q on the sides of the angle. The line through P is perpendicular to the line connecting the feet of Q. Similarly for the line through Q. The midpoint of PQ is the center of a circle that goes through the projections of P, Q on the sides of the angle.

Given two isogonally conjugate points P and P'.

- The projections of the two points onto the sides of the triangle are 6 points on the same circle. The center of the circle is the midpoint of the line PP' . This circle is called the pedal circle of P (or of P'). The 9 pt. circle is the pedal circle of O and H.
- The sides of the pedal triangle of P are perpendicular to the lines drawn from the conjugate point P' to the vertices.



Famous points and their conjugates



The Symmedian point and Lemoine geometry

A parallel to side **BC** of a triangle forms a directly similar triangle to **ABC**, each sharing vertex **A**. The median through **A** goes through the centers of all such parallels.

An antiparallel to side **BC** of a triangle forms an inversely similar triangle to ΔABC . The symmedian through **A** goes through the middle of all such antiparallels.

For a parallel line crossing **AB** at **E** and **AC** at **F**, corresponding angles are equal. So, $\angle B = \angle AEF$. $\Delta AEF \sim \Delta ABC$.

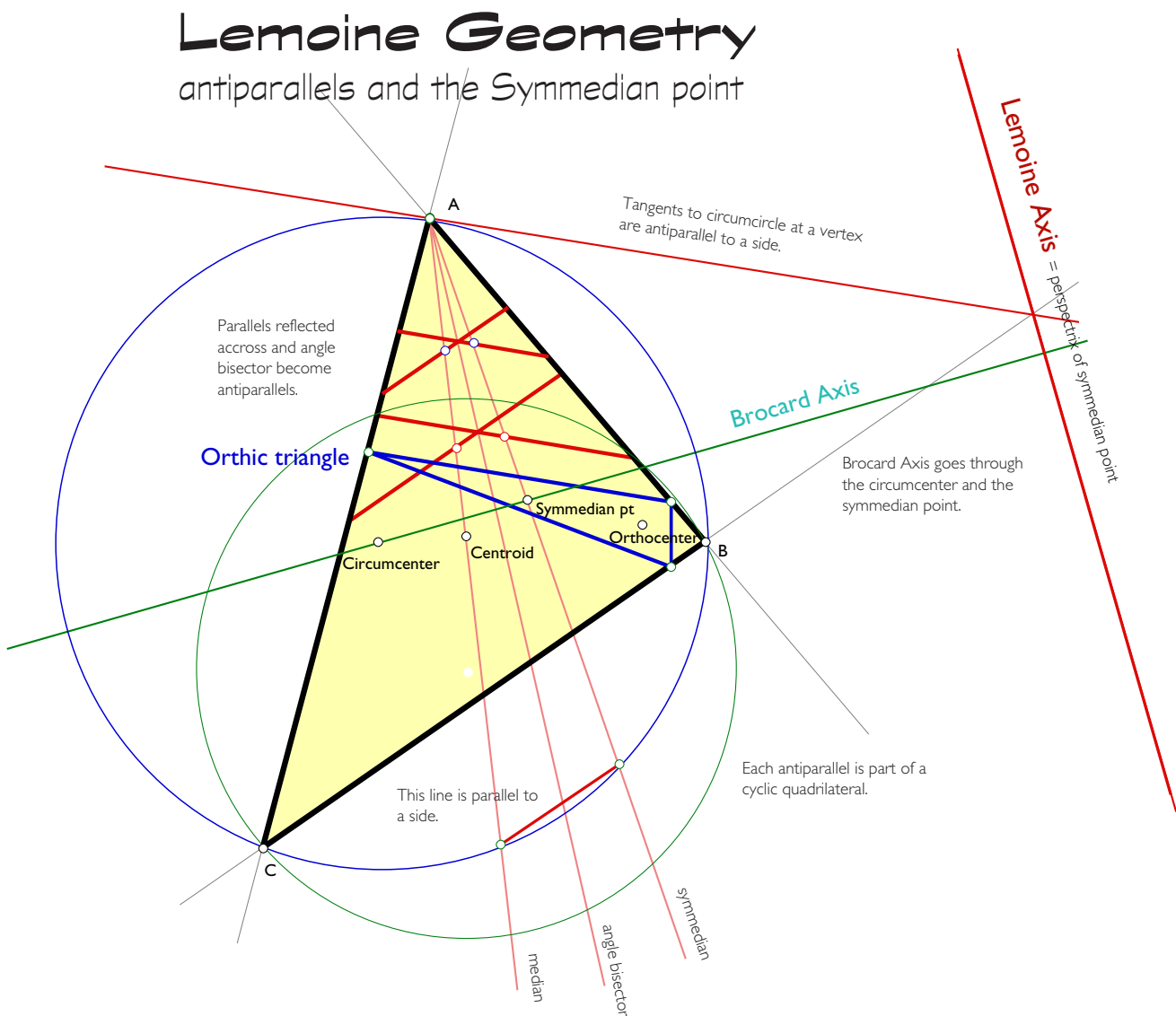
For an antiparallel line crossing **AB** at **E** and **AC** at **F** with $\angle B = \angle AFE$. $\Delta AFE \sim \Delta ABC$. **EF** and **BC** are opposite sides of a cyclic quadrilateral.

If you reflect a parallel across the corresponding bisector, it becomes an antiparallel.

Sides of the medial triangle are parallel to the sides of **ABC**. The exmedians are parallel to the sides of **ABC**.

Sides of the orthic triangle are antiparallel to the corresponding sides of **ABC**. Tangents to the circumcircle at the vertices are antiparallel to the corresponding sides of the triangle.

Notice that antiparallels have slightly more structure associated with them. This leads to a set of theorems complementary to and somewhat independent of the regular geometry of the triangle based on **O**, **H**, **G**, and **I**. This new structure is somewhat loosely called the Lemoine geometry of the triangle.



Coordinates

The two main systems are barycentric coordinates and trilinear coordinates.

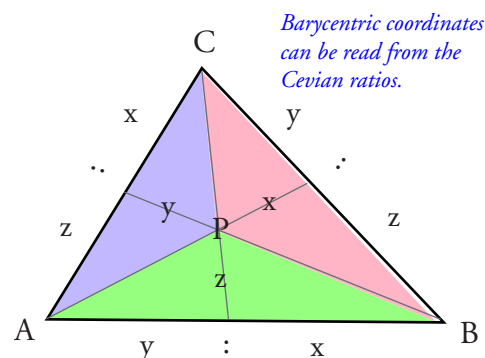
If \mathbf{A} , \mathbf{B} , and \mathbf{C} are the vectors representing the vertices of a triangle, then any point \mathbf{P} in the plane can be written as the vector equation $\mathbf{P} = x \mathbf{A} + y \mathbf{B} + z \mathbf{C}$ with $x + y + z = 1$. (x, y, z) are the barycentric coordinates of \mathbf{P} . If $0 \leq x, y, z \leq 1$, then the point is inside the triangle. As examples $(1, 0, 0)$ are the coordinates of \mathbf{A} . $(t, 1-t, 0)$ represents a point in side \mathbf{AB} of the triangle. The barycentric coordinates of the centroid are $(1/3, 1/3, 1/3)$. An advantage of using three coordinates is that one can refer to each vertex or each side.

The most useful property is that barycentric coordinates can be read from the ratios in Ceva's theorem.

The "mass points" technique is a form of barycentrics.

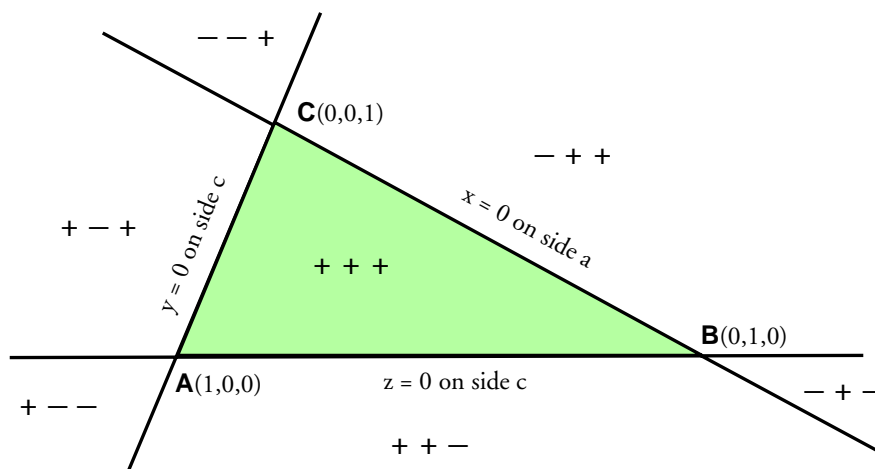
Trilinear coordinates of \mathbf{P} are the three distances of \mathbf{P} from the sides of the triangle. Since I have already sent you all something on trilinears, I do not elaborate farther.

Points in triangles are often described by homogeneous coordinates. When we describe inherently 2D points with 3 coordinates, we are allowed an extra mathematical restriction. Above we called (x, y, z) the coordinates with the restriction $x + y + z = 1$. A second and more common restriction is to make the coordinates accurate only to a ratio. In this case (x, y, z) is considered the same as (kx, ky, kz) where k is any positive constant. Homogeneous coordinates are independent of scale. Since most of the properties we are concerned with are independent of the size of the triangle, homogeneous coordinates are good at expressing many deeper properties of triangles. Homogeneous coordinates often use the notation $(x:y:z)$ to emphasize that ratios are really what matters.



Homogeneous coordinates use the following sign conventions. One coordinate is zero on each side. All coordinates are positive inside the triangle. The diagram shows where the various coordinates are negative.

The signs of the various coordinates. True for trilinear or barycentric coordinates.



Brocard geometry

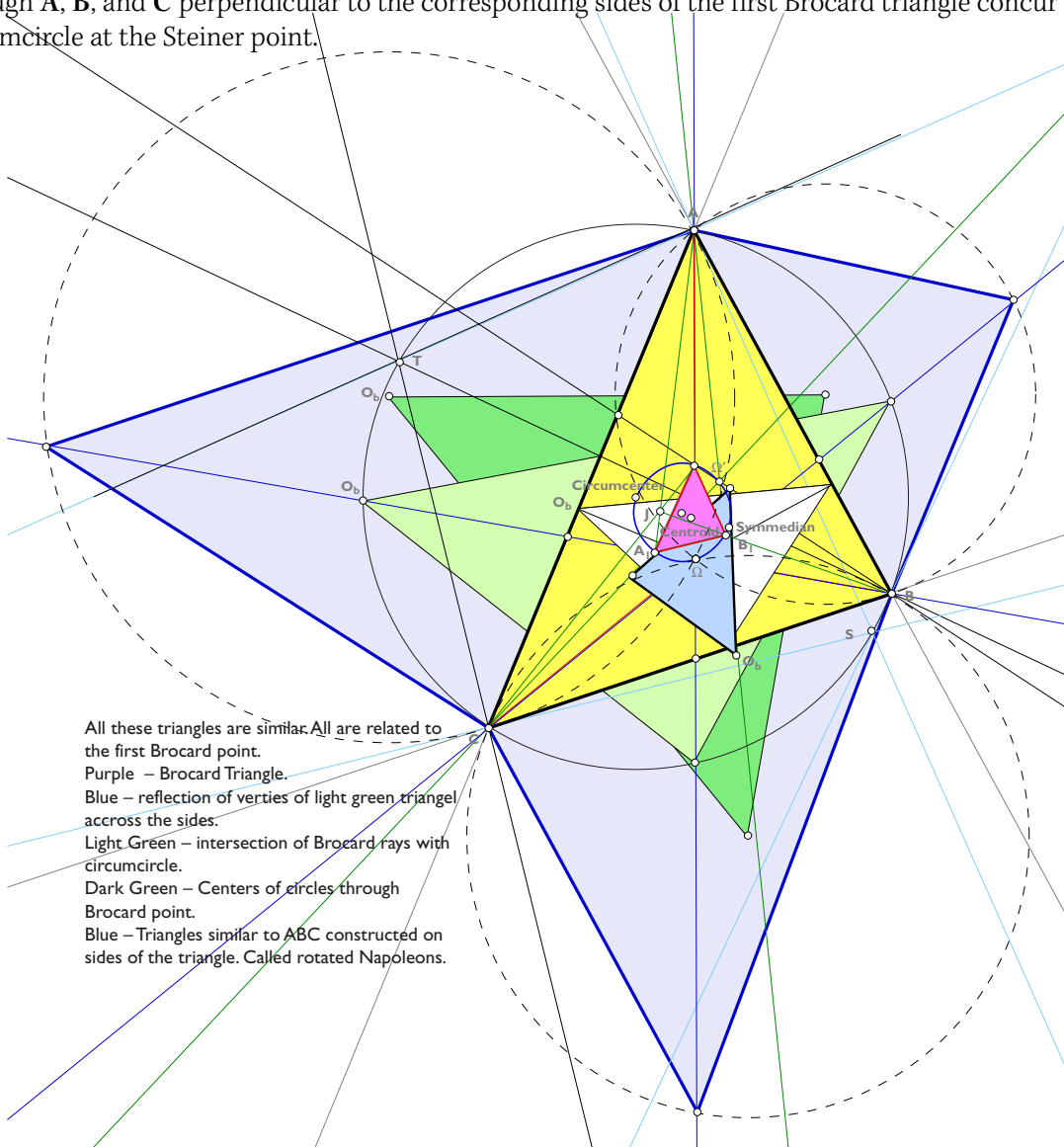
There is a very special triangle inside \mathbf{ABC} whose centroid is the same as the centroid of \mathbf{ABC} and is similar to \mathbf{ABC} . It is called the Brocard triangle. An astonishing number of results are related to it.

There are two points in the triangle for which each cevian makes the same angle with a side. These are the Brocard points. The angle is the Brocard angle and is unique to the shape of the triangle.

The Brocard points are isogonally conjugate. The points $\mathbf{A_1B_1C_1}$ (see picture) form a triangle, the first Brocard triangle. Its circumcircle, the Brocard circle, also goes through the circumcenter \mathbf{O} and the symmedian \mathbf{K} , as well as the two Brocard points.

The pedal triangles drawn from the Brocard points are similar to $\Delta\mathbf{ABC}$. They are the only such pedal triangles. Various other triangles associated with the Brocard configuration are also similar to $\Delta\mathbf{ABC}$ (see picture below). In fact the special uniqueness of the Brocard configuration is the number of the similar triangles it produces.

Lines drawn through \mathbf{A} , \mathbf{B} , and \mathbf{C} parallel to the corresponding sides of the first Brocard triangle concur on the circumcircle at the Steiner point. The Steiner ellipse intersects the circumcircle at this point. Lines drawn through \mathbf{A} , \mathbf{B} , and \mathbf{C} perpendicular to the corresponding sides of the first Brocard triangle concur on the circumcircle at the Steiner point.

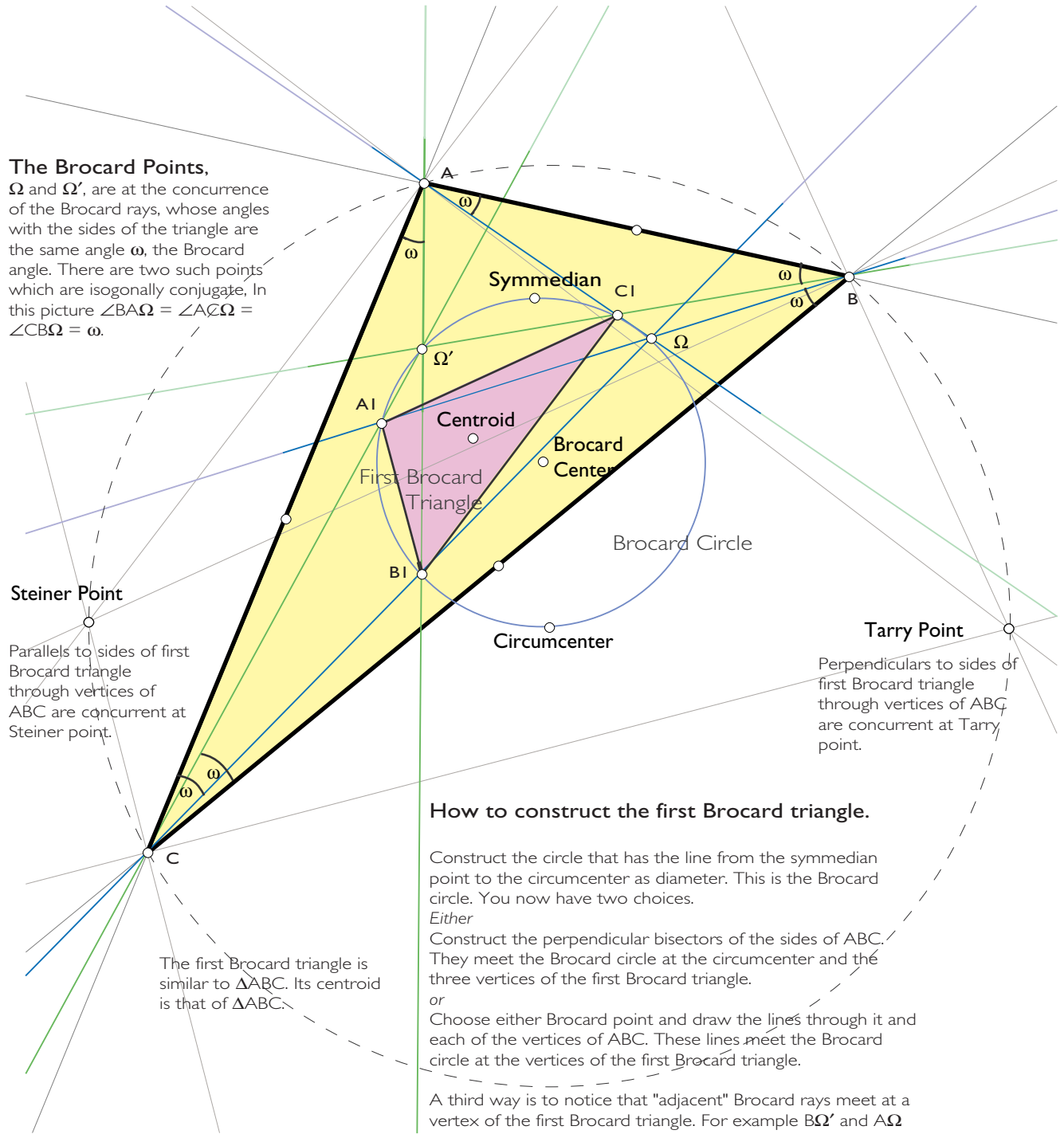


Brocard Geometry of the Triangle

Brocard circle, first Brocard triangle, Steiner and Tarry points

The Brocard Points,

Ω and Ω' are at the concurrence of the Brocard rays, whose angles with the sides of the triangle are the same angle ω , the Brocard angle. There are two such points which are isogonally conjugate. In this picture $\angle B A \Omega = \angle A C \Omega = \angle C B \Omega = \omega$.



Steiner Point

Parallels to sides of first Brocard triangle through vertices of ABC are concurrent at Steiner point.

Tarry Point

Perpendiculars to sides of first Brocard triangle through vertices of ABC are concurrent at Tarry point.

How to construct the first Brocard triangle.

Construct the circle that has the line from the symmedian point to the circumcenter as diameter. This is the Brocard circle. You now have two choices.

Either

Construct the perpendicular bisectors of the sides of ABC. They meet the Brocard circle at the circumcenter and the three vertices of the first Brocard triangle.

or

Choose either Brocard point and draw the lines through it and each of the vertices of ABC. These lines meet the Brocard circle at the vertices of the first Brocard triangle.

A third way is to notice that "adjacent" Brocard rays meet at a vertex of the first Brocard triangle. For example $B\Omega'$ and $A\Omega$ intersect at a vertex.

