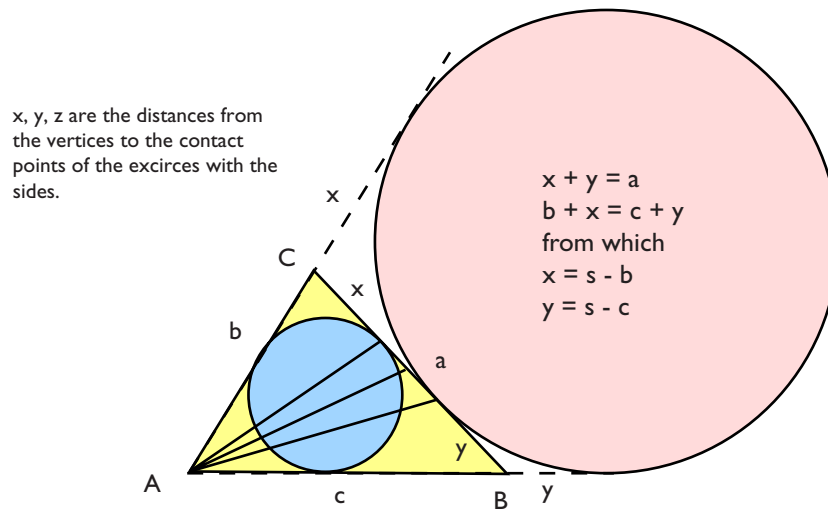


BARYCENTRIC COORDINATES



The three rays in this picture are a Gergonne ray, a median, and a Nagel ray. This picture can be used to show that the Gergonne and Nagel points are isotomes.

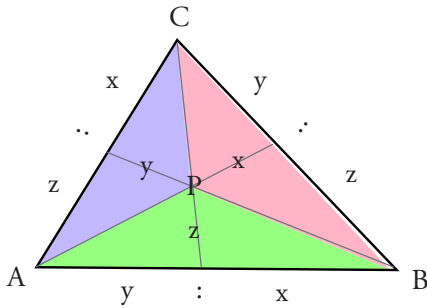
Contents:

- Barycentric coordinates
- The incenter family of points
- Conway on Barycentrics
- Conway compares various coordinate systems

Barycentric Coordinates

Two coordinates to describe a point in the plane. For several reasons it is better to describe points in two dimensional triangle geometry by three coordinates. Triangles are the fundamental shape and tend to see things in threes.

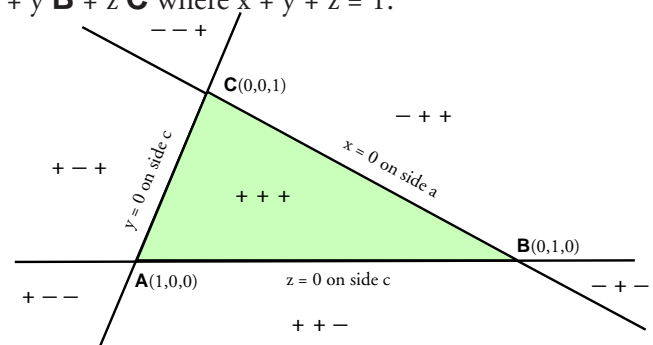
Barycentric coordinates (x, y, z) are such a coordinate system. Since three numbers are used when only two are needed, we add one other condition to these coordinates. If $x + y + z = 1$, we say that the coordinates are normalized. Alternately only ratios are used with $(x, y, z) = (kx, ky, kz)$ and $k > 0$. In this case we write the coordinates as $(x : y : z)$. Coordinates valid up to a ratio are called homogeneous coordinates. Many properties of triangles are independent of scale so ratio independence can be an advantage.



The barycentric coordinates $(x : y : z)$ of a point P can be read from the cevian ratios of P or the areas caused by P (see picture). It is surprisingly easy to find the coordinates this way. If **A**, **B**, and **C** are the vectors representing the vertices of the triangle then the barycentric coordinates are used to write **P** as $\mathbf{P} = x \mathbf{A} + y \mathbf{B} + z \mathbf{C}$ where $x + y + z = 1$.

The vertices are $A(1, 0, 0)$, $B(0, 1, 0)$, and $C(0, 0, 1)$.

The signs of the coordinates are shown in the diagram to the right. The x coordinate's sign depends on its position relative to side a. It is not possible for all three coordinates to be negative.



Computations in homogeneous coordinates are a bit unusual since we are free to multiply the coordinates by any factor. As an example $(bc : ac : ab) = (1/a : 1/b : 1/c)$ since we can divide each coordinate by the factor abc .

Often coordinates are symmetrically expressed in terms of the sides or angles of the triangle. For example the coordinates of the Nagel point are $(b+c-a : a+c-b : a+b-c)$. In this case a shorthand notation can be used $(: a+c-b :)$, showing only the middle coordinate.

The importance of symmetric forms: symmetric expressions in the angles or sides of a triangle have special importance. The area, the semiperimeter, and the radii of the inscribed and circumcircles are all symmetric in the sides of the triangle. Using the ratio independence of the coordinates the incenter $(: b :)$ can be written in a second form as follows $(: b :) = (: b/2R :) = (: \sin B :)$, where R is the circumradius.

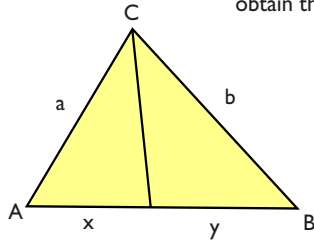
The centroid

As an example of the use of barycentric coordinates, we can see that the coordinates of G, the centroid, are $(1:1:1)$. This is seen since the centroid is the concurrence of the medians, which divide each side equally in a 1:1 ratio.

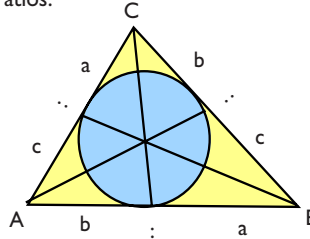
Barycentrics of the Incenter family of points

The Incenter

An angle bisector satisfies the formula $a/b = x/y$.



Applying this relationship to all three bisectors we obtain these ratios.



We read the barycentric coordinates from the cevian ratios.

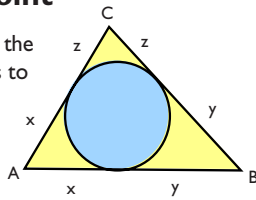
Incenter coordinates
 $(a : b : c) = (: b :)$

This final form is Conway's shorthand notation. For points where the coordinates have the same form only the middle coordinate is shown.

The Gergonne Point

The Gergonne point G_o is the concurrence of the cevians to the contact points of the incircle with the sides.

We first find some properties of the triangle with its incircle. Let $x, y,$ and z be the distances from the vertices of $\triangle ABC$ to the contact points of the incircle with the triangle.



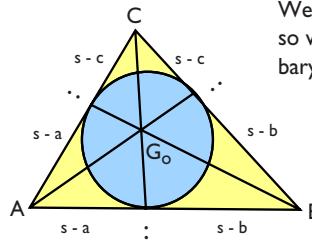
$$2x + 2y + 2z = \text{perimeter}$$

$$x + y + z = s$$

$$x = s - (y + z)$$

$$x = s - a$$

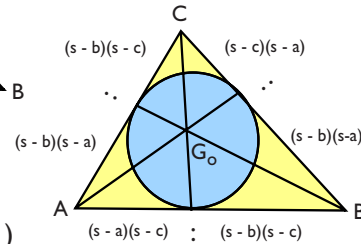
We use the shorthand $sb = s - b$.



We adjust the cevian ratios so we can see the barycentric coordinates.

Gergonne point coordinates
 $(sb \ sc : sa \ sc : sa \ sb) =$
 $(1/sa : 1/sb : 1/sc) = (: 1/sb :)$

Divide all coordinates by $sa \ sb \ sc$ to get this form.

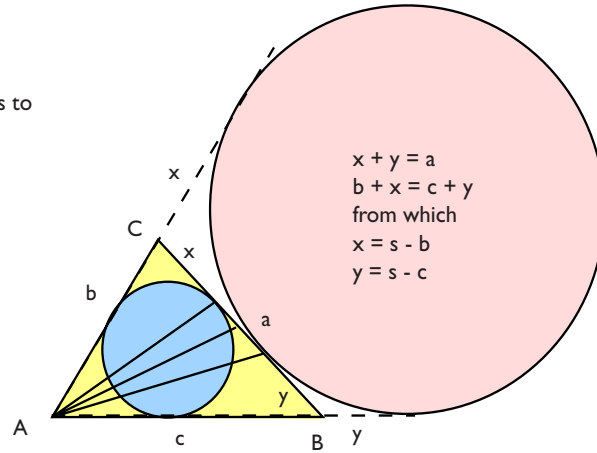


The Nagel Point

The Nagel point N_o is the concurrence of the cevians to the contact points of the excircle with $\triangle ABC$.

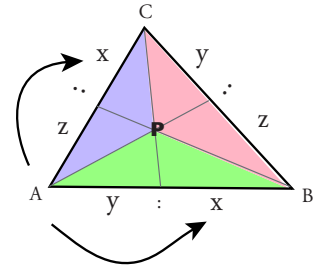
Nagel point coordinates
 $(: sb :)$

x and y are the distances from the vertices to the contact points of the excircles with the sides.



The three rays in this picture are a Gergonne ray, a median, and a Nagel ray. This picture can be used to show that the Gergonne and Nagel points are isotomes.

How to read the barycentric ratios from measurements on the triangle. Coordinate x corresponds to vertex A. Notice the position of the x 's compared to their vertex. The y and z coordinates are equivalently placed. Remembering that we are free to adjust ratios by multiplying both elements by the same number, we adjust the ratios so that the barycentric coordinates can be read.



Date: 12/2/98 12:27 PM
 From: John Conway, conway@math.Princeton.EDU
 To: Steve Sigur, ssigur@netdepot.com
 CC: geometry-college@forum.swarthmore.edu

Unfortunately I read few introductory books myself, and the only ones I know that use barycentrics are very old and in any case, say very little. But there is in fact very little that needs to be said!

The NORMALIZED barycentrics of P with respect to A,B,C are the unique numbers (X,Y,Z) adding up to 1 for which the vector equation

$$P = XA + YB + ZC$$

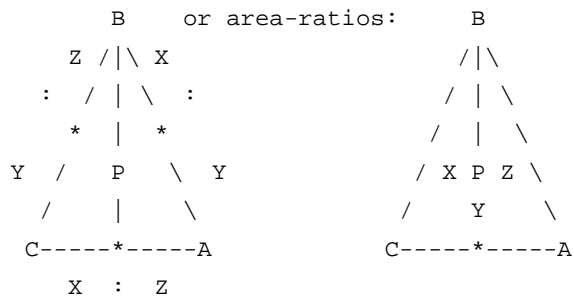
holds - UNNORMALIZED ones are any triple proportional to these. For the latter I write (X:Y:Z) with colons to indicate that only their ratios matter.

Since you are familiar with orthogonal trilinears, which I write [x,y,z] or [x:y:z], you'll need to know the conversion rules

$$\begin{aligned} [x:y:z] &\text{ converts to } (ax:by:cz) \\ (X:Y:Z) &\text{ converts to } [X/a:Y/b:Z/c]. \end{aligned}$$

So there's very little difference, but the difference is valuable!

If you draw your triangles the way round I do, X:Y:Z are easily read off from the Cevian ratios:



(sorry I can't draw the separating lines CP,AP).

So plainly $G = (1:1:1)$, or, when normalized $(1/3,1/3,1/3)$, meaning that $G = (A+B+C)/3$ in vector terms.

The subordinate and superior of P are

$$3G/2 - P/2 \quad \text{and} \quad 3G-2P \quad (\text{normalized})$$

or

$$(Y+Z:Z+X:X+Y) \quad \text{and} \quad (-X+Y+Z:X-Y+Z:X+Y-Z) \quad (\text{unnormalized}) \quad \text{as we see by using } G = (X+Y+Z:X+Y+Z:X+Y+Z).$$

The isotomic and isogonal conjugates of P are

$$(aa/X:bb/Y:cc/Z) \quad (\text{conjugal})$$

and

$$(1/X:1/Y:1/Z) \quad (\text{isotome}).$$

Sometimes, of course, the OT formula is the simpler one. For instance the conjugal of $[x:y:z] = (ax:by:cz)$ is $(aa/ax:bb/by:cc/cz) = (a/x:b/y:c/z) = [1/x:1/y:1/z]$, simpler than the barycentric formula, while its isotome is $[1/aax:1/bby:1/ccz]$ (more complex).

But THIS simplicity obscures the situation. The isotomic conjugate is an affinely invariant concept (in other words, if you linearly stretch the triangle, this relation remains the same) - we can see this because the barycentric formula $(1/X:1/Y:1/Z)$ mentions no special constants like a,b,c . The isogonal conjugal isn't affinely invariant (it depends on angles); so its barycentric formula SHOULD mention the shape of the triangle.

If you look upstairs, you'll see that the formulae for subordinates and superiors are also constant-free in barycentrics, again because these are affinely invariant - again that is obscured in OTs.

Another instance of the same phenomenon is that

$G = (1:1:1)$ in barycentrics, but $[1/a:1/b:1/c]$ in OTs,
while

$I = (a:b:c)$ in barycentrics, but $[1:1:1]$ in OTs.

Again, the simplicity of G in barycentrics tells us something - that the centroid is an affinely invariant concept - while that of I in OTs tells us nothing.

The switch to barycentrics should be quite painless if at first you just get into the habit of calling the typical point $[X/a:Y/b:Z/c]$ rather than $[x,y,z]$ - in other words, regard the barycentrics just as renamed trilinears. As I said, they're very much the same, but the slight difference is important. The numbers a,b,c are mere nuisance-factors that clutter up most of the OT theory and often obscure the situation.

John Conway

I'm sending this to several people interested in triangles, and hope they'll confirm receipt (except for rkg, who might be out of email touch), and send any further substantial messages about triangles to everyone else on the list.

There are three standard systems that use three coordinates to represent a point in the plane of a give triangle, namely

BARYCENTRICS (B), AREALS (A), and ORTHOGONAL TRILINEARS (OT).

Often the last is shortened to "TRILINEARS", but I prefer the longer name since in fact all three systems are trilinear. Each may or may not be normalized. For the not-necessarily normalized versions I'll use:

$(X:Y:Z)$ for B and A, and $[x:y:z]$ for OT

and for the normalized ones

(X,Y,Z) for B, and $[x,y,z]$ for OT.

All three (unnormalized) systems are very similar - indeed they coincide for Barycentrics and Areal, so I'll usually call these jointly BA, and the conversion from TO to these is very simple:

$[x:y:z]$ becomes $(ax:by:cz)$.

So for many purposes it hardly matters which system one uses. However, there ARE ways in which one or other of the systems is better than another, and it is the purpose of this note to point out that when one takes all these into account the barycentric system emerges as the clear winner.

The letters B,A,BA,OT before each numbered point below show how this decision was reached. A letter N indicates that normalized coordinates are involved.

OT 0N. The distances of P from the sides are

$$2X.\Delta/a, 2Y.\Delta/b, 2Z.\Delta/c \quad [x, y, z].$$

A 1N. The areas of PBC, PCA, PAB are the normalised areal coordinates

$$X.\Delta, Y.\Delta, Z.\Delta \quad [ax/2, by/2, cz/2]$$

B 2N. If VA, VB, VC are vectors to A, B, C, then $P = X.VA + Y.VB + Z.VC$

[Of course these are just the definitions of the three systems.]

BA 3. The Cevian ratios are $Y:Z, Z:X, X:Y$ [$by:cz, cz:ax, ax:by$].

BA 4. The Menelean ratios of the lines

$$PX + QY + RZ = 0 \quad \text{and} \quad px + qy + rz = 0$$

are

$$-Q:R \quad -R:P \quad -P:Q \quad \text{and} \quad -cq:br \quad -ar:cp \quad -bp:aq$$

B 5. The normalizing condition is

$$X + Y + Z = 1 \text{ (B) or } \Delta \text{ (A), } [ax + by + cz = 2\Delta]$$

OT 6. The isogonal conjugate (or "conjugal") is

$$\text{co-P} = (a^2/X : b^2/Y : c^2/Z) \quad [1/X : 1/Y : 1/Z]$$

BA 7. The isotomic conjugate (or "isotome") is

$$\text{iso-P} = (1/X : 1/Y : 1/Z) \quad [1/(x.a^2) : 1/(y.b^2) : 1/(z.c^2)]$$

B 8. AFFINE INVARIANCE. If an affine transformation takes

$$A, B, C, \text{ and } P = (X,Y,Z) \text{ to } A_1, B_1, C_1 \text{ and } P_1,$$

then the barycentric coordinates of P_1 with respect to the new triangle $A_1 B_1 C_1$ are still (X,Y,Z) .

(It is because isotomic conjugation is an affinely invariant concept that its expression (see #7) in barycentrics cannot involve the edge lengths of the triangle.)

BA 9. The concepts of subordinate and superior points. These points are the images of P in the subordinate (or "medial") triangle, whose vertices are the midpoints of the edges of ABC, and the superior (or "anticomplementary") triangle, the midpoints of whose edges are A,B,C. We have:

$$\text{sub-P} = (Y+Z : Z+X : X+Y) \quad [(by+cz)/a : (cz+ax)/b : (ax+by)/c]$$

$$\text{super-P} = (Y+Z-X : Z+X-Y : X+Y-Z) \quad [(by+cz-ax)/a : (cz+ax-by)/b : (ax+by-cz)/c]$$

(Again the simplicity of the BA coordinates is due to affine invariance.)

B 10. RATIONALITY. X,Y,Z are rational functions of the

Euclidean coordinates of the points A,B,C,P. If P is

a point that's rationally defined from A,B,C, then its barycentric coordinates are rational functions of a^2, b^2, c^2 . This is true, for instance, of the centroid, orthocenter, circumcenter, symmedian point, Brocard points, and so on.

(This is an extremely important point, and extends to give the very useful property below.)

BA 11. ALGEBRAIC CONJUGATES. Many points (such as the incenter) can be obtained by solving algebraic equations that have other (algebraically conjugate) solutions. Passing to these other solutions then yields further points that have essentially the same geometric properties (in this way, we get from the incenter to the excenters). We can get the barycentric coordinates of such "companions" as the appropriate algebraic conjugates of those of the original.

The simplest case of this is when the coordinates are rational functions of a, b, c but not of a^2, b^2, c^2 . So for example if a point that's rationally constructed from the incenter has barycentric coordinates

$$(X(a,b,c), Y(a,b,c), Z(a,b,c))$$

then the corresponding point obtained from the a -excenter is simply obtained by "changing the sign of a ", thus:

$$(X(-a,b,c), Y(a,-b,c), Z(a,b,-c)).$$

For example the Nagel point is the "super-incenter" $(b+c-a : c+a-b : a+b-c)$, and so its a -companion is $(b+c+a : c-a-b : -a+b-c)$.

The OT coordinates of these points are much harder to understand:

$$[b/a + c/a - 1 : c/b + a/b - 1 : a/c + b/c - 1]$$

and

$$[b/a + c/a + 1 : c/b - a/b - 1 : -a/c + b/c - 1].$$

Well, that will do for now. I'll just survey the "winners"

0	1	2	3	4	5	6	7	8	9	10	11
OT	A	B	BA	BA	B	OT	BA	B	BA	B	BA

In only two cases is OT the winner, and in all other cases but one (the definition of A!) B is at least a joint winner.

I have deliberately preferred conceptual reasons for preferring one system to another, rather than mere comparisons of the simplicity of the coordinates for particular points. Some points are simpler

under one system rather than another, and often it's OT that would give the simpler ones. But this difference can never be great, since $[x,y,z]$ translates to $[ax,by,cz]$; and in barycentrics the simplicity

often has a useful conceptual meaning.

For example $(1:1:1) = [1/a : 1/b : 1/c]$ is the centroid, and its simplicity in barycentrics comes from its affine invariance.

On the other hand $(a:b:c) = [1:1:1]$ is the incenter, more complicated in barycentrics since it has algebraic conjugates $(-a:b:c)$ $(a:-b:c)$ $(a:b:-c)$. The apparent simplicity in trilinears disappears when we pass to the sub-incenter (Spieker point)

$$(b+c:c+a:a+b) = ((b+c)/a : (c+a)/b : (a+b)/c).$$

The "Morley perspectors" look simpler in OT:

$$[\cos(A/3) : \cos(B/3) : \cos(C/3)] \text{ and } [\sec(A/3) : \sec(B/3) : \sec(C/3)]$$

but again this apparent simplicity disappears when we want a bit more:

the Barycentric versions

$$(a.\cos(A/3) : \dots) \text{ and } (a.\sec(A/3) : \dots)$$

can be conjugated to get all the "companion Morley perspectors" just as easily.

In summary, the simplicity of coordinates for particular points can go either way, and in any case is not a strong argument. It's their theoretical properties that make barycentrics the clear winner.

John Conway