

REFLECTIONS ON DUALITY

by Michael Matthews

Duality changes context when it is considered in different dimensions: in three, duals are created by connecting centers of 2D faces (squares in cube), while in four dimensions a dual is usually created by connecting centers of 3D cells (cubes in hypercube). I want to continue discussing duality in both dimensions here.

UNFOLDED DUALITY

It can be difficult to get a good look at what is really going on when dealing with polytopes in three and four dimensions. An interesting strategy I stumbled upon is one that deals with examining duality by looking at the figures fold out. I will start with a cube because of its level of intuitive quality. A cube fold out will have six squares aligned in the cross pattern. The dual to the cube will have a vertex at the center of each face, so we mark those with points. When we connect the centers of the squares to create the dual (just the five squares first), we get four right triangles connected in the plane sharing a common vertex [Fig. 1].

This is very interesting. We know that the dual to the cube is the octahedron, which is 1) eight equilateral triangles, and 2) two “diamond” based pyramids sharing the same base diamond (note: diamond and square are same thing in 2 space, and in this case diamond is more appropriate). So we can look at our four right triangles as 1) half of the octahedron, the triangles of which will become equilateral when the squares are folded into a cube, or 2) a perspective view looking into the bottom half of an octahedron. When we add the last square with its centered vertex, four more triangles will be created [Fig. 2] (which are difficult to represent in the plane) by connecting to the four outer squares (just like the inside square did).

We can now do this for other three dimensional polyhedra, but I'd rather move directly to the hypercube instead (haven't checked out the other 3D polyhedra yet, but would assume similar results would be achieved). The folded out hypercube will have 8 cubes aligned in a very similar cross pattern with one internal cube that is “opposite” the last external cube. The dual to the hypercube will have a vertex at the center of each cell in the hypercube, so we mark those points. When we connect the centers of the cubes to create the dual (just the seven cells first), we get eight right tetrahedra sharing the same vertex, which is the center point of the “inside” cube [fig. 3].

We know from experience (and from “Beyond the Third Dimension”) that the dual to the hypercube is the 16-cell, which is i) a composition of 16 regular tetrahedrons. But we know now, from both analogy and process, that the 16-cell is also ii) two octahedrons sharing the same “base surface” (base surface being the outer crust / shell / armor of the octahedron, analogous to the “base diamond” before, which was the outer edges of the “diamond based pyramid” base diamond). So we can look at our 8 right tetrahedra as i) half of the 16-cell, the tetrahedra of which will become

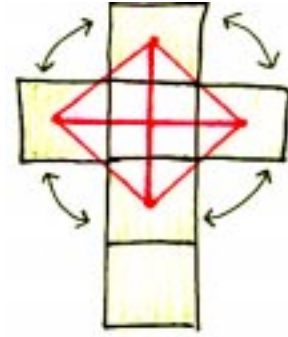


fig 1

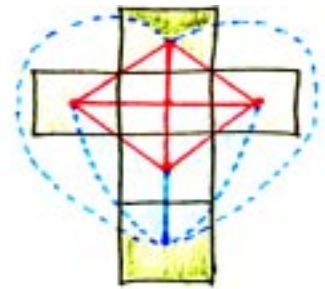
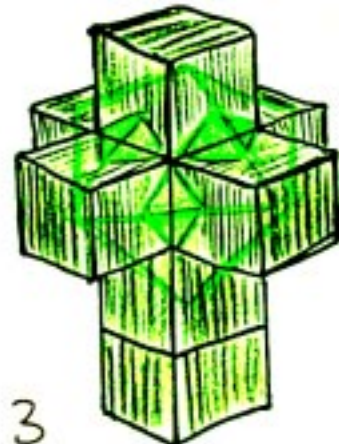


fig 2



fig 3



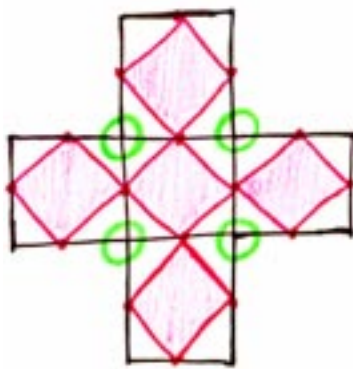
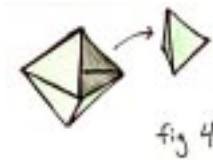
regular when the cubes are folded into a hypercube, or ii) a skewed perspective view looking into the bottom half of a 16-cell. When we add the last cube with its centered vertex, 8 more tetrahedra will be created by connecting to the eight triangles on the outside of the “inner octahedron”.

OTHER TYPES OF DUALITY

A. THREE SPACE

In three space another type of polyhedral duality exists, creating irregular polyhedra (polyhedra composed of more than one type of regular two dimensional polygon) rather than regular ones.

This type of duality involves connecting the center of edges rather than faces [fig 5]. Take note of the interesting way that this type of duality includes the original type of duality in a dimension one less than the one we’re working in. I will begin with the cube again. Originally I drew a cube, marked the centers of edges and connected them. I then realized that in the process I was doing two things: i) creating a new shape out of each original face, and ii) creating a new shape by effectively “shaving” or slicing off the corners (vertices). For a cube, this process transforms each square into its “dual” diamond and each corner into a triangle (we sliced off a regular tetrahedron). Thus, our edge-dual polytope has 6 regular diamonds and 8 regular tetrahedron. *The faces became their dual, and the corners became whatever one got by slicing half way through an edge.*



Note that another way of getting the same result, using a technique we’ve already used before, is unfolding the cube and looking at the “raw material” of our new irregular polyhedra [fig 6]. Each square has its dual, and each triangle has been separated and lies flat in the plane like a square (marked by the green circles). Notice how the center of these “separated areas” are the corners of the cube. When the squares are folded up, any line of the inner diamond will act as the base for the triangles in our irregular polyhedra (the other lines of the triangles created out of the other diamonds). This will all come back when we check out 4D polytopes.

A table will hopefully be sufficient for the rest of the three dimensional polytopes:

Faces (become)	Corners (become)	Product:
Tetrahedron: 4 faces -> 4 triangles	4 corners -> 4 triangles	Octahedron
Cube: 6 faces -> 6 diamonds	8 corners -> 8 triangles	14 sided solid
Octagon: 8 faces -> 8 triangles	6 corners -> 6 diamonds	14 sided similar solid as above
Dodecahedron: 12 faces -> 12 pentagons	20 corners -> 20 triangles	32 sided solid
Icosahedron: 20 faces -> 20 triangles	12 corners -> 12 pentagons	32 sided similar solid to above

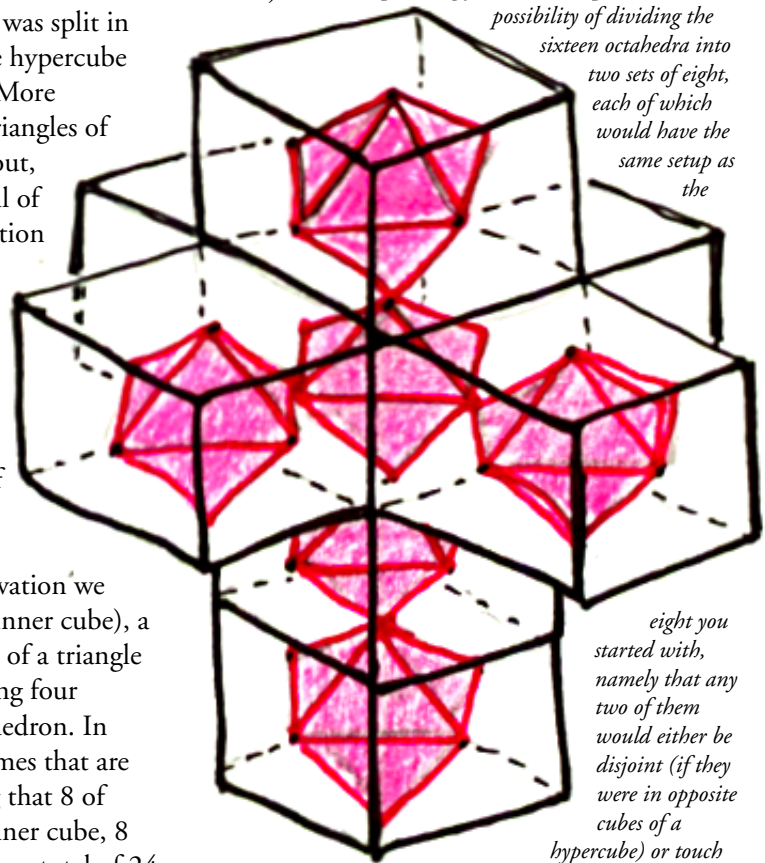
Now a few questions are raised. 1) what is the relationship of the “similar solids”? 2) what happens if you connect the centers of the faces of the new irregular polyhedra? More duality right?

B. FOUR SPACE

In four space more types of polyhedral duality exist, creating analogous irregular polytopes. These type of dualities involve connecting the center of the 2D faces and 1D edges of 4D polytopes rather than centers of the 3D cells. Once again, our original concept of duality, in this case three dimensional duality, appears in these new duality concepts. My work has only just begun, so I've only examined the new type of duality that is created by connecting the centers of the "faces" (2D objects) of a 4D polytope. I will begin with the hypercube. Because I can't just draw the hypercube, I start with the unfolded hypercube to get a look at the "raw materials" just like we did when we unfolded the cube. Working by analogy, as well as by actually connecting the points, we see that each cell (cube) is transformed into its dual octahedron (the analog of the diamond).

So we know we have 8 octahedra, six surrounding an inner one with an extra outer one at the bottom. Just as we had "spaces that aren't really spaces" before when we unfolded the cube, here we have volumes that are not really volumes. These volumes occur at the corners of the inner cube where, just as the third edge from the unfolded cube's vertex was split in two in our 3D example, our fourth edge at the hypercube vertex is seen in this fold out split three ways. More analogy, just as the spaces before became the triangles of our 3D irregular when we folded up our fold out, here our volumes will become some type of cell of our 4D irregular when we fold it up. The question is what type of cell will the fold up create?

Notice that each corner of the inner cube of the foldout is associated with one triangular face of its octahedron. These triangular faces will act as the "base surfaces" of our 3D cells when we fold the cubes into 4D, connecting the separated edges / points, just as the lines of the inner diamond in part A acted as the bases to what became the triangular faces of our 3D irregular polyhedron. So by analogy and observation we know that we will have (for one corner of the inner cube), a triangle base, each edge of which will be a part of a triangle created by the "trisected hypercube edge", giving four triangles aligned exactly as they are in an octahedron. In fact, when the cubes are folded up these "volumes that are not really volumes" become octahedra. Noting that 8 of these will be formed from the corners of the inner cube, 8 more from the corners of the outer cube, giving a total of 24 identical octahedra (we already had 8 from the original cube duals), which is commonly know as the 24-cell! As a check, the volume of an octahedron (for a cube edge length of $2x$) is $8x^3/3$, which is $1/3$ total volume of the cube. So 24 octahedra (total volume $64x^3$) equals total volume of 8 cubes.



Response from Prof. Banchoff.

Your discussion of the edge-truncation of the hypercube is really nice, and you have done it in a different way than I did in B3D, where the approach was more in terms of coordinates, in Chapter 8. By analyzing things synthetically, beginning with the cuboctahedron in three-space, you have come up with an excellent way of visualizing the twenty-four octahedra, eight of them easy to see and sixteen others that are more obscure. The way the corner octahedra are presented as four pieces fitting together, in analogy with the three pieces of triangle fitting together to give a triangle in the cuboctahedron, is especially instructive. One thing that might be worth pursuing further at this point is the

possibility of dividing the sixteen octahedra into two sets of eight, each of which would have the same setup as the

eight you started with, namely that any two of them would either be disjoint (if they were in opposite cubes of a

hypercube) or touch at a single vertex (if they were in adjacent cubes). This fits in with an idea I heard at a conference in Canada two summers ago where it was observed that the 96 edges of the 24-cell could be split up into three sets of 32, each being the set of edges of a hypercube. I'll bet you come up with that if you continue your investigation in this way.

An easy way to come to this conclusion theoretically rather than through sketches, now that we have already “proved” it by looking at the actual raw materials, is take the two rules we used for creating our three dimensional irregular polyhedra: i) faces (or cells in this case) become their duals, and ii) the corners become whatever you get when you “shave” half way to the other one. So now it is all made very simple: the 8 cells become 8 octahedra, and the 16 vertices of hypercube become 16 octahedra (p. 49 B3D), giving a total of 24 octahedra.

I’ve only just begun to look at 4D duality created by connecting the midpoint of edges (32 of them in the case of the hypercube). Using the same process as above, imagine the unfolded hypercube, each of the 8 cubes already “shaved” into the irregular polyhedra as described in part A before (figure5), folding together. Each vertex would become a tetrahedron (so 16 of them), and the final 4D irregular polytope would consist of these 16 tetrahedra and the original 8 irregular polyhedra. More later possibly

Check out David Akers stuff

Response from Prof. B.

These are the polyhedra that we will call “semi-regular” in keeping with commonly accepted usage. Just as a figure is called “regular” only if it has regular polygon faces and additional symmetry constraints at the vertices, we will require that the semi-regular objects have symmetry too, so that every vertex looks like every other vertex, and moreover, so that there is a symmetry of all of space taking the object to itself taking any specified vertex to any other vertex position. The second condition is required in order to rule out some objects that “locally” have the same shape, but are rather asymmetric in other ways.

A key example here is similar to a polyhedron we began discussing a while ago and never completed. We started with a cube and began shaving off the twelve edges, leaving squares (not diamonds) in the center of each face of the cube. We cut the corners off too, giving in general eight equilateral triangles, six squares, and twelve congruent rectangles. By cutting off just the right amount, the rectangles will also be squares and we will have a semi-regular figure with the same symmetries as the original cube, with eighteen squares and eight triangles.

Now consider the following modification of this surface. We take the top square and its adjacent four squares and four triangles, and lift this “cap” up, then turn it forty-five degrees, and place it back down. The resulting figure will not have as much symmetry as the cube, but it will have at every vertex three squares and a triangle, and every vertex will have the same shape as every other. This figure was found early in this century by a Russian mathematician who claimed that he had found a semi-regular figure to add to those already enumerated by Archimedes. Either you accept that as a new discovery, or you go back and change the definition so that only figures with the appropriate symmetry need apply.

In any case, as you examine these semi-regular polyhedra, you should rediscover Archimedes’ classification for convex polyhedra in three-space, and see how many of these figures will have analogues in higher dimensions.

Good work. I look forward to seeing such things in your final project. Also good use of sketches, in the spirit of Stringham. We can also develop some fiord figures to accompany the final document (although I would like to see the sketches preserved as well).