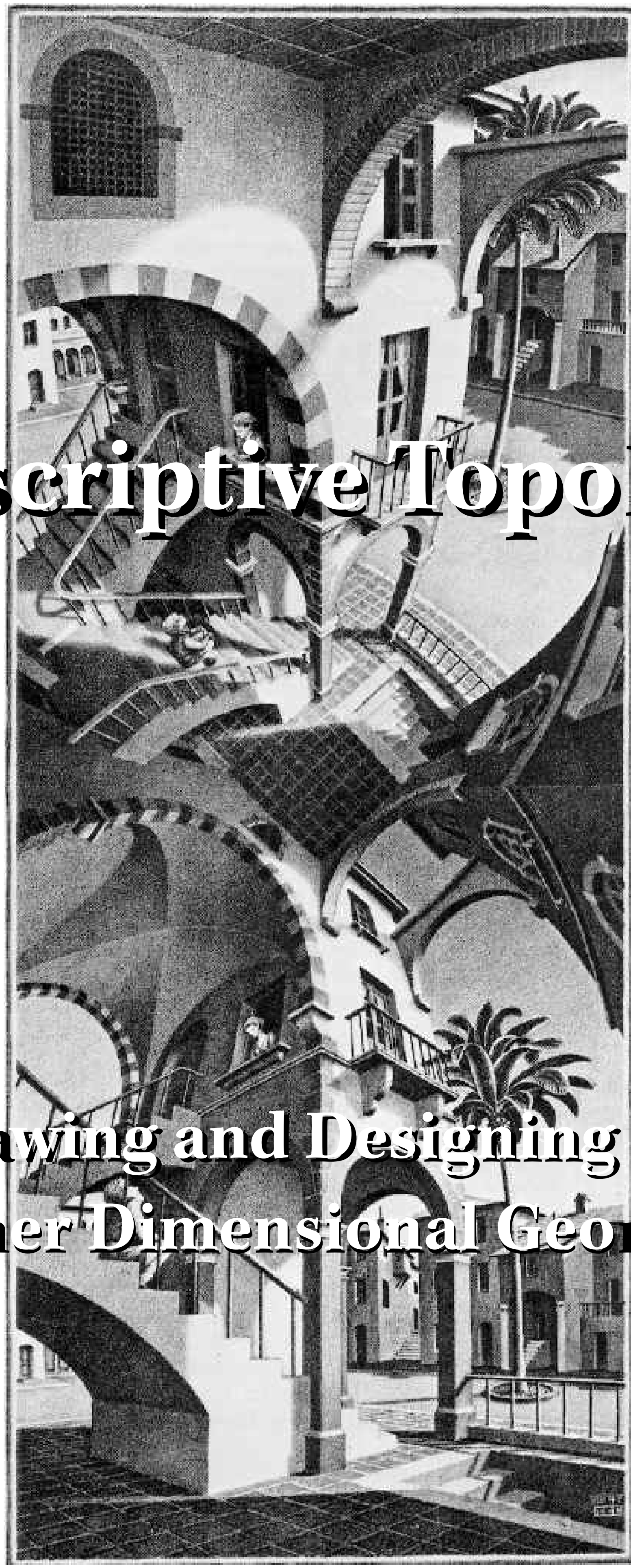
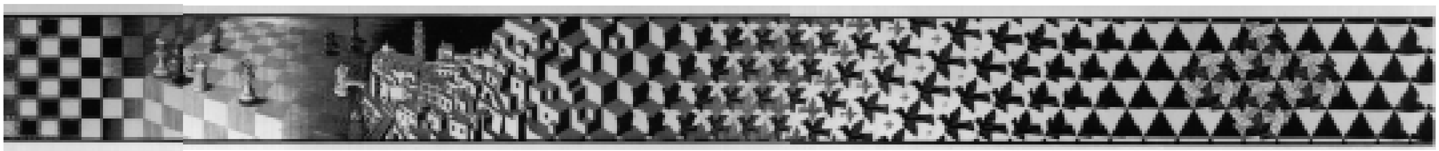
Descriptive Topology

Drawing and Designing into Higher Dimensional Geometry





"Awareness of the three dimensions, the notion of plasticity, is not as general as one might expect in spatial creatures like ourselves. An understanding of the relationships between plane & space is a source of emotion for me; and emotion is a strong incentive, or at least a stimulus for making a picture."

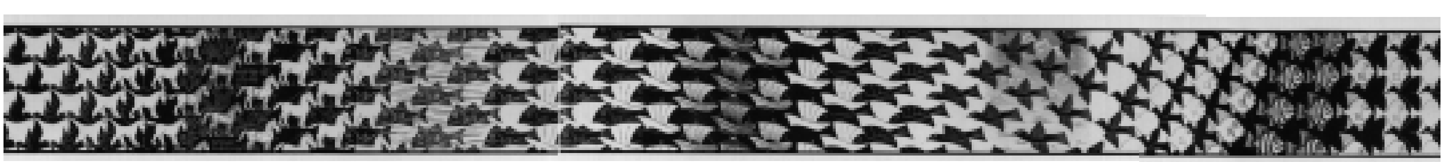
"If you want to express something impossible, you must keep to certain rules…. The element of mystery to which you want to draw attention should be surrounded and veiled by a quite obvious readily recognizable commonness."

"An artist's talent is not only determined by the quality of the thoughts he wishes to convey — for anyone can have the most beautiful, the most moving images in his head — but also by his ability to express them in such a way that they get through to other people, undistorted. the result of the struggle between thought & the ability to express it, between dream & reality, is seldom more than a compromise or an approximation."

-M.C. Escher

"We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time."

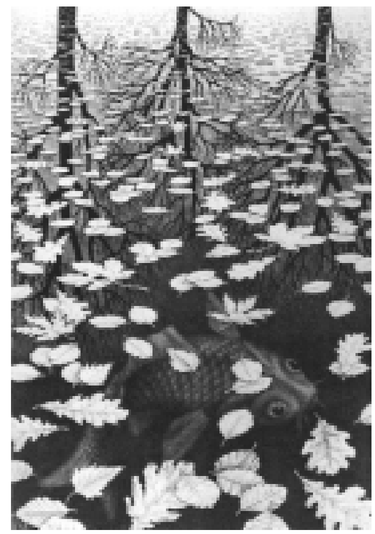
-T.S.Eliot
Little Gidding



Approaching the fourth dimension…cautiously

Most people don't spend a great lot of their time considering how the world spread before their senses is a place both strange & beautiful. The complexity of the world around all of us, the collection of sensations & perceptions, is essentially limitless and ever-changing. And yet so many minds function as if all the sensations, all the changes, were merely predictable parts of a common, uninteresting universe of which they are a central figure. They do not stop to notice the everyday patterns, chaotic as the movement of running water or orderly as a crystalline solid of ice.

These forms are as beautiful as strange and exist in every dimension of our behaviors & experiences, from the use of language down to the basic perception of space. But when looking at any one of these forms closely enough, one must dissociate those things that are *explicit* from those things that must be *inferred* by our interpreting brain. Does language really mean something, or does our thinking mind have to learn which ideas are attached to which succession of sounds? Do we really perceive a three-dimensional world, or do we make some fairly basic assumptions about the behavior of objects around us? And which of these assumptions can we actively suppress when we want to expand our horizons?



The fourth dimension is still rather mysterious to me, but not simply as the next unconquered realm of spatial perception — the one immense step above our measly egocentric senses. In trying to understand the behavior of a four-dimensional environment, we find ourselves reevaluating the ways that we understand things in three dimensions. As in all journeys outside the realm of commonness, we must learn how to keep track of our assumptions & lose them when they become cumbersome. If such an approach is taken with a measure of caution, one always stand to learn a great deal not just about the destination but even more about the place of origin.

The first step I took toward this uncommon territory was through drawing — and from that perspective it never seemed all *that* unusual. Drawing is partly about conveying properties of dimensions in pictures, perhaps even three-dimensional configurations & geometries with shadows & lighting, but usually along the flat surface of paper. In searching for clever ways to draw things & to represent images, one who draws or designs is bound to find links between lower & higher dimensions, art & geometry, and if you're paying attention, algebra & calculus as well. A well-constructed drawing *can* tell you more about a 3-D function than five pages of endless algebraic calculation. *If* it's done well.

Understanding how to draw function graphs, surfaces, objects in 2-D & 3-D, and gaining a deeper insight into how we perceive our spatial world in general: these things are crucial before making any premature leaps into higher dimensions.

A bit more on our mode of spatial perception...

When we look at a picture of a 2-D object, we have no trouble imagining its “2D” features — connected lines, angles between the lines, maybe even vertices. Of course, we have no trouble seeing that the page with the picture on it is a 3-D object, with curvature & shadows. We also can easily construct a mental model of our three-dimensional environment in order to, say, find the piece of paper in the dark.

But clearly such a spatial sense is not all visual. In fact, vision is not all that good at perceiving in three dimensions — certainly not compared with the tactile system, for which the sensory apparatus itself is three-dimensional. Unlike every other sensory organ in the body, skin can conform to quite complex three-dimensional configurations, and with the aid of body movements, can *actively* sense its surroundings. The sensing takes place in direct contact with the three-dimensional objects, so the tactile sensation already comes with depth information in it.

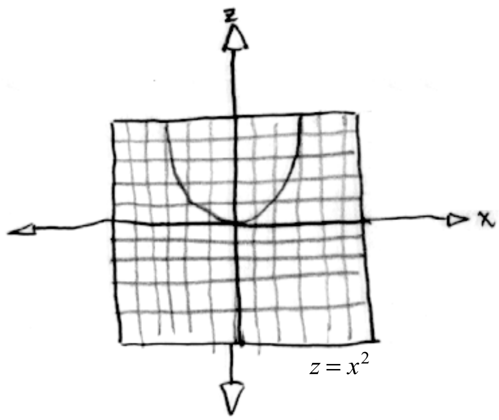
Our treasured visual system, on the other hand, is allotted only two small disks of neural tissue (retinae) affixed to the backs of gel spheres — not all that useful for actively probing the environment, and by no means an ideal tool for recognizing the subtle qualities of depth & texture. And yet even without the aid of other senses, our visual systems can quickly pick out depth information from even a complex and moving environment.

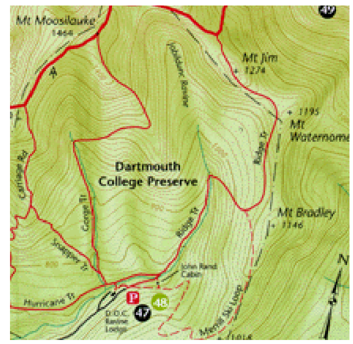
In the absence of other sensory cues, our brains have learned to *infer* depth information from the signals provided by each eye. The signals first meet in the brain, where their differences can be interpreted in terms of depth cues. The result of this interpretation is an initial sketch of the surroundings that is certainly more than two-dimensional, but in effect only provides 3-D information about the sides of objects that are facing us (this is sometimes called a 2›-D sketch). Our brains have learned to pick out cues from such a sketch to infer the full 3-D structure of surfaces & objects. Other tools like memory serve to fill in gaps when we turn off a light and must assume that our three-dimensional world has not changed.

This isn’t so tough for our brains in a 3-D world, where we have come to expect the environment to act in common 3-D fashion. We can learn all manner of rules of thumb to distinguish a shadow from a dark object in a corner, to tell that a car is not getting bigger but is indeed coming closer. Learning to expect & predict patterns of behavior is the essence of gaining an intuition about the structure underlying that behavior. Nevertheless, our vision ultimately relies on well-learned rules of thumb.

In order to forge such an intuitive understanding of higher dimensions, we must start by making inferences from two and three dimensions. We must find simple examples of four-dimensional objects that behave like their lower dimensional analogs. We can’t learn about a four-dimensional cube (or *hypercube*) by taking hold of it in our hand. So we must instead come up with rules of thumb for visually recognizing the entirely ‘other’ sense of depth, into & from which the fourth dimension extends.

Drawing function graphs





Drawing the graph of a function is not the same thing as plotting points between two axes & connecting the lines. Plotting can yield sufficient visual information for some simple functions, most of which contain fewer than three variables. Even for simple functions, however, there is often something interesting about the math that could easily be communicated visually — i.e. by drawing an accurate picture or set of pictures. For instance, a function as simple as $z = x^2$ can be plotted easily enough, but unless the plotter has an eye for the symmetry across the x-axis in this equation, he or she may disregard the visual symmetry required to make it accurate. If you intend to learn something while drawing a function, whatever the dimension, take the time to make it accurate; or at the very least, take the time to know which parts of the graph are interesting enough to draw well.

Once you get to drawing graphs of three variables, plotting is no longer a viable option. You must resort to drawing with an eye for esthetic details, if you want graphs to be accurate. However, in 3-D you have far more interesting options for rendering & representation:

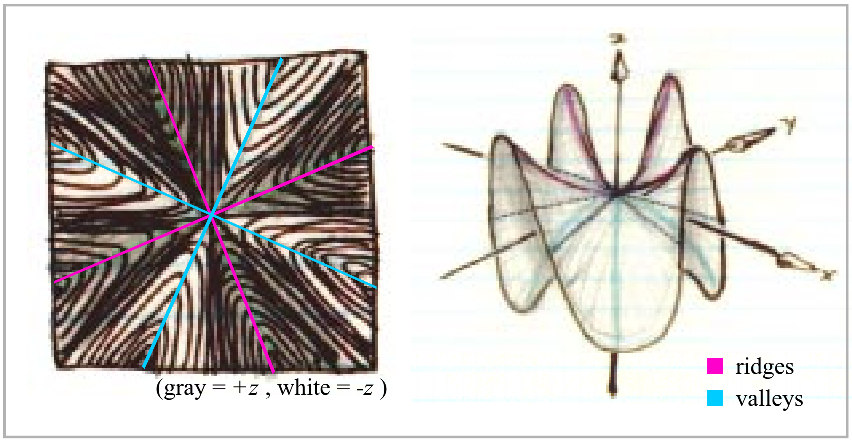
Topographic mapping is a way that many graphic artists concerned with representing 3-D surfaces (namely, the cartographers of the world) make complicated surfaces much easier to fathom. These maps are usually composed of what are known as *level sets*, since each line represents a set of points on a surface that are level with one another. Because the level sets in topographic maps are drawn at regular depth intervals (every 20 ft. in the map below), the distance between the lines carries depth information (lines closer together mean steeper climbing). Level sets can also be drawn *onto* a 3-D surface rendering, showing the set of values at a certain height, like $z = 0$.

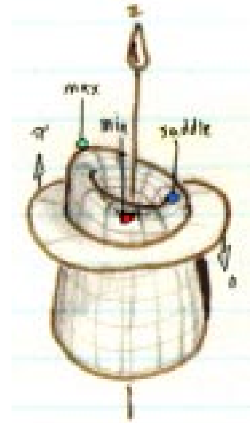
A topographic map is usually rendered from a viewpoint directly above a surface, where the *contours* in the surface must be inferred from the spacing of the lines. But contours can be made explicit by drawing surfaces from viewpoints that draw attention to interesting features. As shown on the next page, the graph of $4xy(x^2 - y^2) = z$ is interpretable in level sets, but far more interesting as a 3-D rendered surface.

Just to mention a few graphic tools that provide some assistance in 3-D rendering:

Drawing axes & a coordinate frame that don't interfere with the properties that you want to highlight in a graph

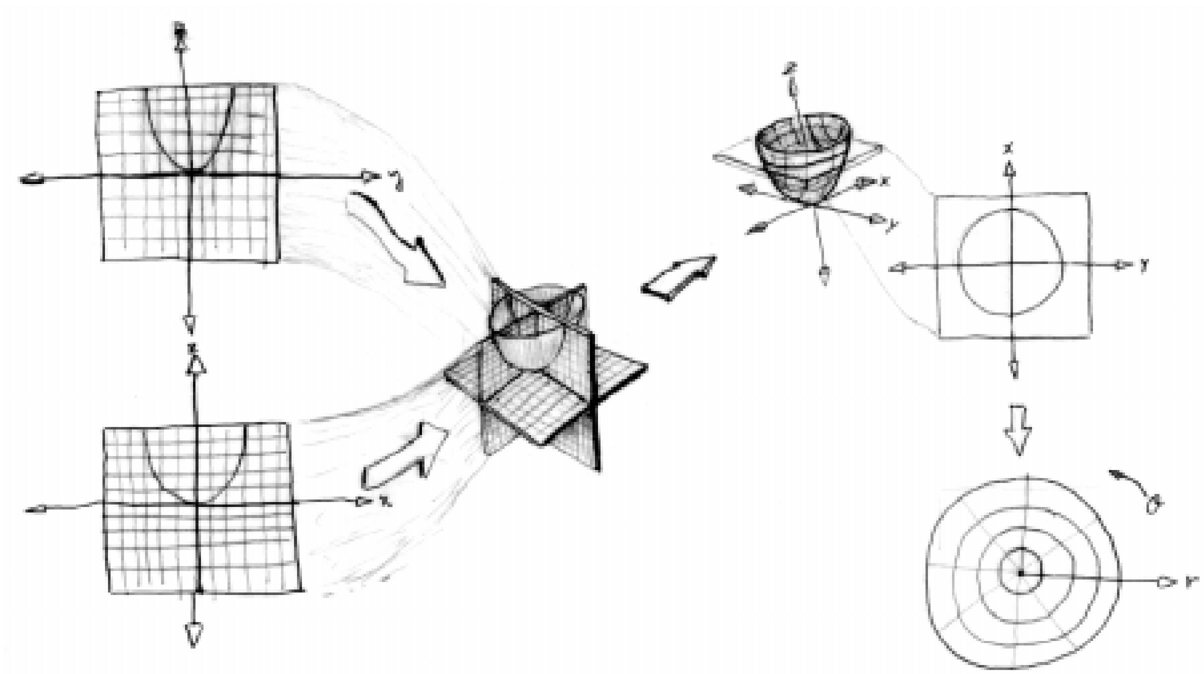
Using graphical aids like arrows & labels





Level sets & 3-D rendering of $4xy(x^2 - y^2) = z$

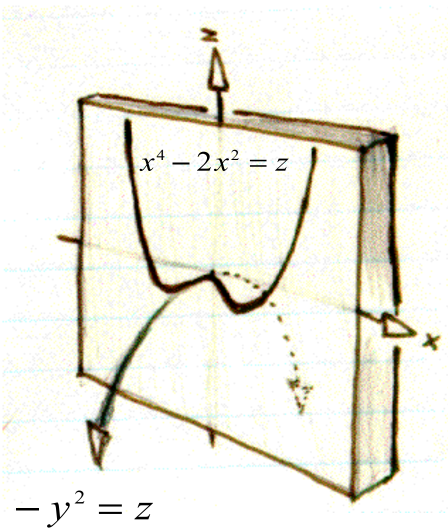
Shading to accentuate depth information, especially when a function has interesting curvature
Of particular interest in any three-dimensional graph is the set of *critical points* — points like global & local maxima, minima, as well as stranger points like saddle points, shoe points & other anomalies.
Using color, even just varying shades of gray, can be quite effective when used sparingly and for the purpose of making something more apparent in a complicated 3-D picture.



The graph of $z = x^2+y^2$ can be seen as a combination of two independent functions: $z = x^2$ and $z = y^2$. Although a drawing inside a set of coordinate planes might seem more accurate, it is when we remove all but the axes that the function appears as a bowl with radial symmetry — a paraboloid. By taking x-y slices at various heights, we can see that the graph is a circle at every level. Such a function can thus easily be transformed into *polar* or *cylindrical* coordinates.

Slicing techniques

Before trying to render a three-dimensional graph, it is always useful to see what individual slices of the surface look like. By setting one variable to a constant value & drawing the 2-D graph of the remaining two variables, you can begin to grasp how a given 2-D slice changes as you move along the third axis.



A simple example is the graph of

$$x^4 - 4x^2 - y^2 = z$$

in which *x* and *y* contribute to the z value independently.

The graph in the xz plane is dominated by an upward parabola (x^4) when x is large and a downward parabola ($-4x^2$) when *x* is small, resulting in the symmetrical 'bump' around *x*=0.

The contribution of $-y^2$ serves simply to pull the "W"-shaped curve along the path of a downward parabola.

Once we understand these 2-D components' roles in the 3-D function, it is not difficult to render the 3-D graph retaining this "slice information". In the graph on the right, blue & pink lines have been added to inform the viewer of the independent *x* and *y* components.

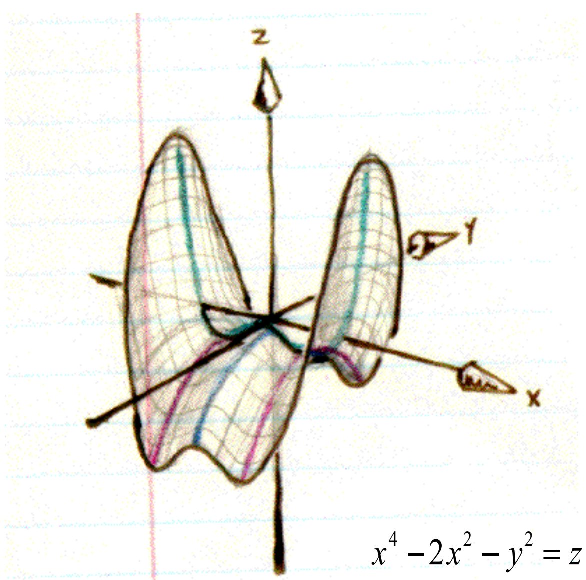
Certain functions can be graphed in a *polar coordinate system*, exchanging two cartesian variables (usually *x* & *y*) for *r* (radius from origin) & θ (angle around a circle in radians).

When using polar coordinates it is a good idea to keep some basic equations in mind:

$$x^2 + y^2 = r^2 \qquad \cos^2\theta + \sin^2\theta = 1$$

$$x = r\cos\theta \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$y = r\sin\theta \qquad 2\cos\theta\sin\theta = \sin 2\theta$$



An introduction to polar or cylindrical coordinates

Transforming a function into another coordinate system can simplify its equation, making its graph far easier to draw. For example, consider the problem of drawing the function:

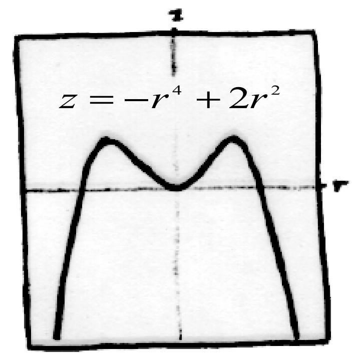
$$z = -(x^2 + y^2)^2 + 2(x^2 + y^2)^2$$

With Cartesian variables (*x, y, z*), this equation is a bit hard to understand. But we can transform the function into a *cylindrical* coordinate frame by replacing every *x* with *r cos* θ and *y* with *r sin* θ (where *r* = *radius* and θ = *angle around a circle*):

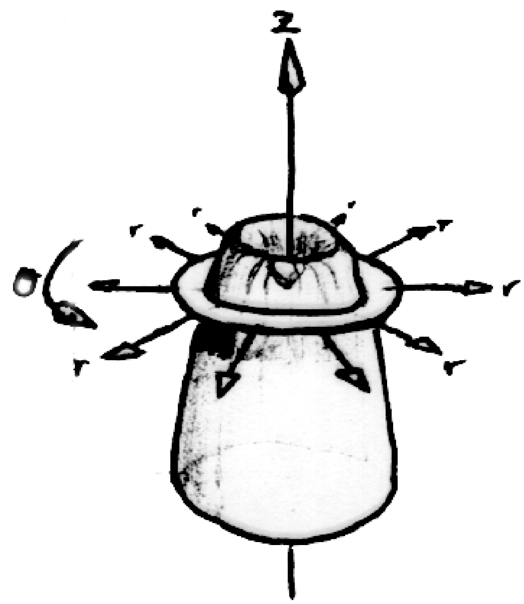
$$z = -(r^2 \cos^2\theta + r^2 \sin^2\theta)^2 + 2(r^2 \cos^2\theta + r^2 \sin^2\theta)$$

and since $\sin^2\theta + \cos^2\theta = 1$, this readily becomes:

$$z = -r^4 + 2r^2 .$$



In this radial system, *z* is a function of only a single variable: the radius, *r*. Thus, we can take a slice through the graph at any angle θ and see the exact same double-peaked graph. Since θ has 'fallen out of the equation,' we can thus conclude that the 3-D graph is radially symmetrical (looks the same at all angles).



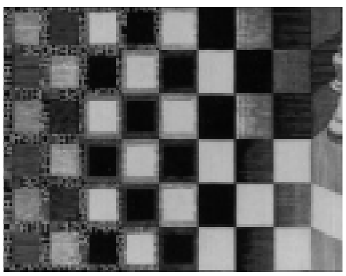
Keep in mind that the two equations (one using *Cartesian* and one using *cylindrical* variables) are equivalent to one another and have the same 3-D graphs. One just happens to be simpler than the other and is therefore more useful when trying to draw it.

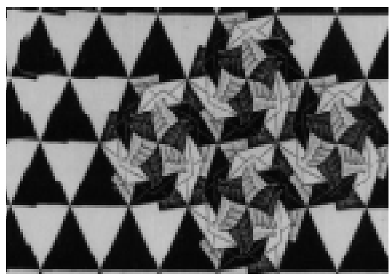
Polygons & Polyhedra

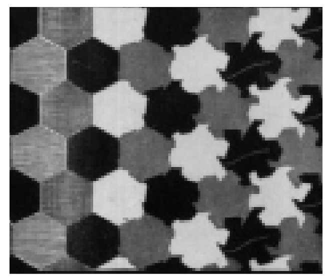
When you start talking about dimensions & space, it quickly becomes convenient to talk about things with finite boundaries, regular forms that are the quintessential objects of any dimension. Regular polygons & polyhedra, like triangles and cubes, are for the most part familiar objects, and yet they can be used in clever ways to solve sophisticated mathematical problems. These regular forms distill certain spatial relationships & symmetries, creating objects that can seem to resonate with an esthetic buried deep within our genetic makeup.

Regular polygons	k	θ (deg)
triangle	3	60
square	4	90
pentagon	5	108
hexagon	6	120
heptagon	7	128.6

It would not be an exaggeration to say that M.C. Escher was obsessed with polygons & polyhedra, and sought endlessly for new ways of using them in his drawings. Of particular fascination to Escher was the division of planes into regular repeating forms. In the pictures below, he used squares, equilateral triangles & hexagons to fill

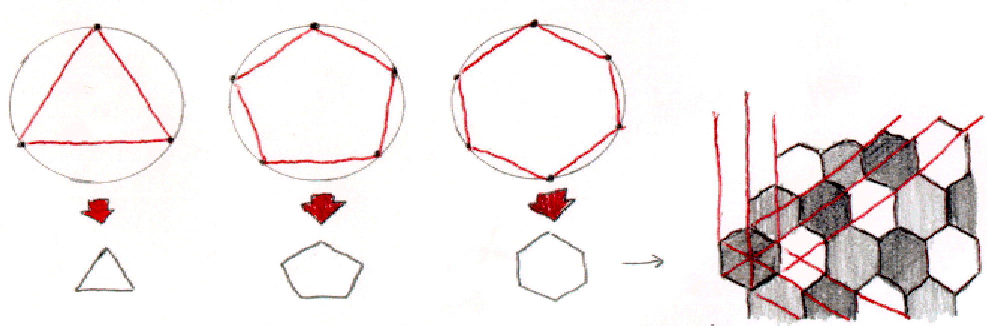






planar surfaces. Are there any other choices when filling planes? Are there reasons that only certain shapes will work?

Drawing polygons can be very simple & accurate by relying on a combination of circles and straight lines. A polygon with any number of sides can be created by evenly spacing dots on a circle then connecting the dots. This can be exact, and it can also lead to a more intuitive drawing approach. Once you begin to pay attention to the symmetries in each polygon, the various shapes start to reveal both their uniqueness & their relationships with other shapes.

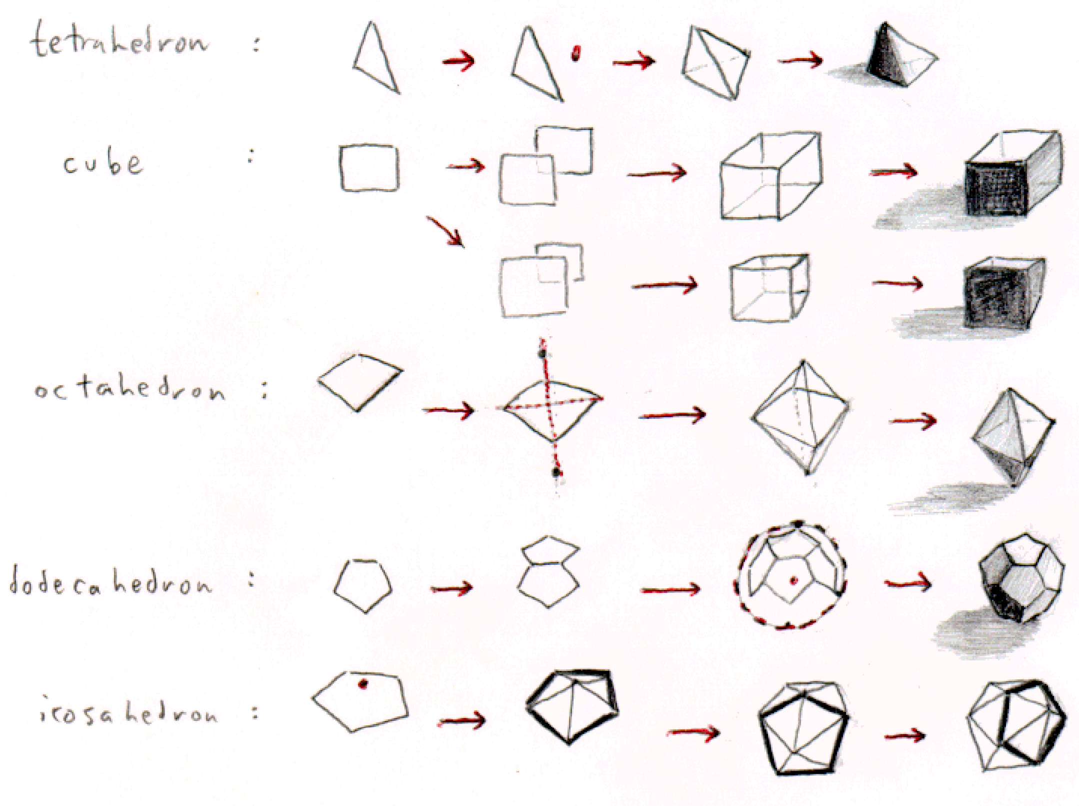


Polyhedra extend the patterns of symmetry & mathematical beauty into three dimensions, where the number of interacting spatial relationships increases exponentially. In the jump to three dimensions, polyhedra gain the freedom to have more sides than vertices (a luxury the regular polygons do not have), and also may take on opacity, viewpoint perspectives, and shadows when they are rendered on a page of paper.

Regular polyhedra	***v***	***e***	***f***	***p***	***q***
tetrahedron	4	6	4	3	3
cube	8	12	6	4	3
octahedron	5	12	8	3	4
dodecahedron	6	30	12	5	3
icosahedron	12	30	20	3	5

v = *vertices*
e = *edges*
f = *faces*
p = *edges at face*
q = *edges at vertex*

For the most part, polyhedra can be drawn by connecting polygons to one another along their edges. Like polygons, polyhedra could be constructed by connecting points spaced evenly around the surface of a sphere. So when putting the pieces of a polyhedron together in a picture, don't forget to keep within certain boundaries & use the basic symmetries that give each object its peculiar nature.



Just draw a triangle, a dot & some lines to connect them.

Connect two squares into a hexagonal projection of a cube

or

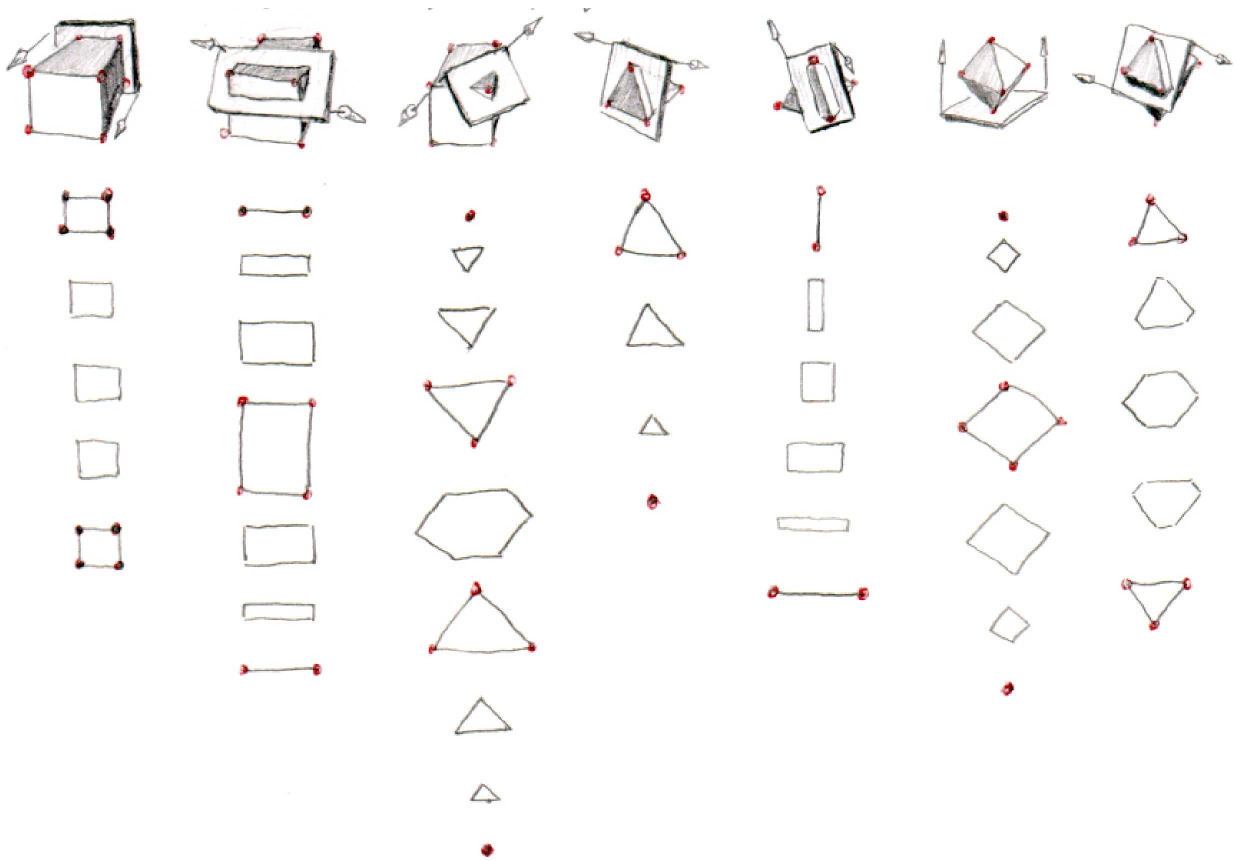
add linear perspective by making one square smaller.

Find two points 'above' & 'below' the center point in a square and connect the vertices.

With the central pentagon as an anchor, add five squished pentagons into an almost circular configuration.

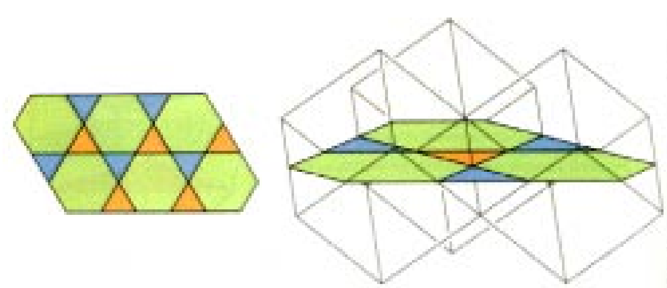
Begin with a first pentagon lying on its side with a point near one of its edges. Make five triangles by connecting all vertices to the point. A second pentagon can now be drawn using two of these triangles. A third pentagon adds the final two equilateral triangles to the picture.

Just as one can slice through three-dimensional contour surfaces, one can draw slices through polyhedra to learn more about how they relate to lower-dimensional shapes and to each other. Slicing a cube face-first, edge-first and corner-first each yield unique depictions of a cube's symmetry & structure. Sometimes squares, sometimes other quadrilaterals, sometimes triangles and very rarely a regular hexagon emerge as slices of the cube. When we slice through tetrahedrons and octahedrons, we come upon the same set of shapes that emerged from the cube. What does this commonality imply about the relationship between the objects?



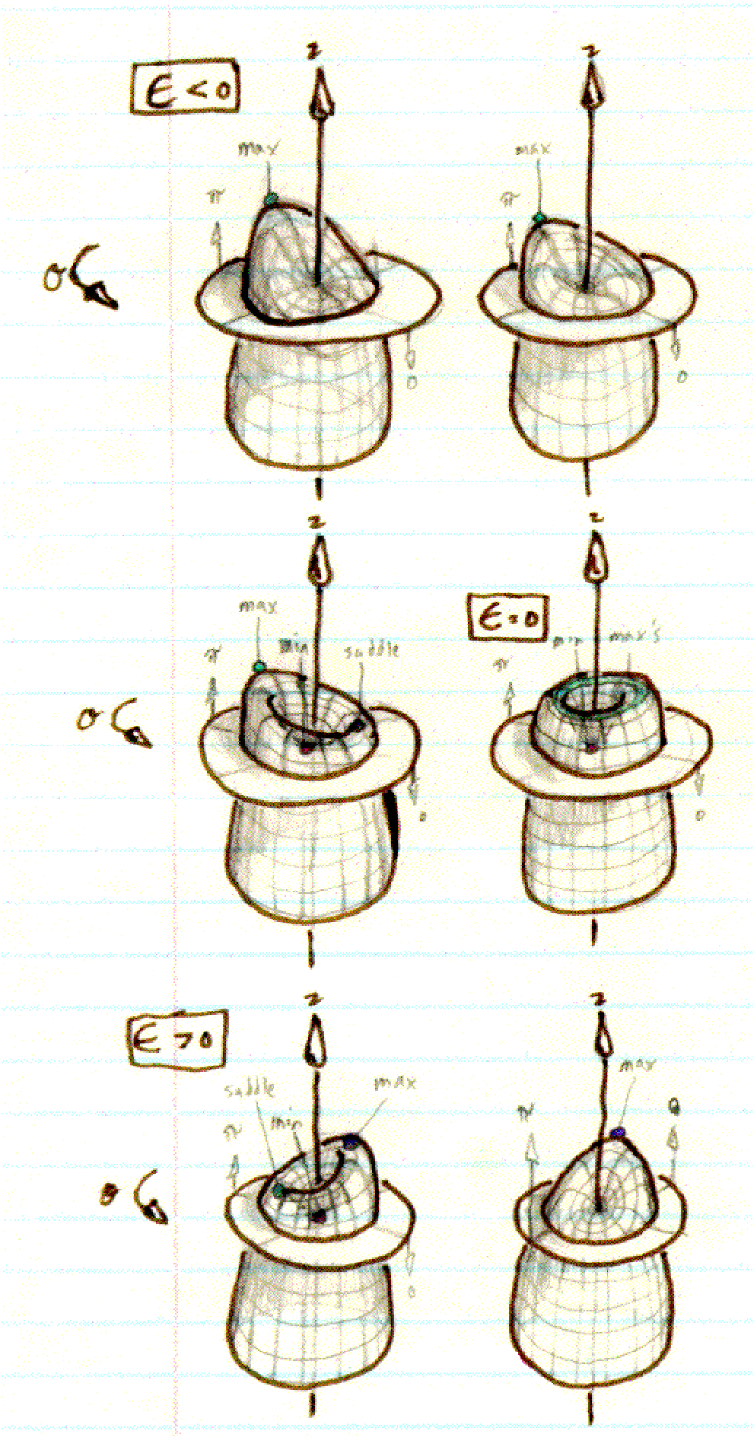
You may have noticed that the set of shapes we get slicing cubes & octahedra is the same set that can be used to divide up the plane with repeating shapes. There is an easy way to see the reason why: imagine a set of cubes stacked into a box such that they are all contiguous and fill the box to the top. Any slice that cuts through the entire box must cut through a set of cubes, and since the cubes are packed together, the slice will be composed of a plane-filling pattern: either hexagons, triangles or quadrilaterals. And no others.

Although tetrahedra & octahedra can also be packed into a box, they would still yield more of the same types of patterns (with slightly different restrictions on how the shapes would fit together).



The pattern of triangles and hexagons in a slice through a collection of cubes (from Banchoff's Beyond the Third Dimension)

Representing higher dimensions



$z = -(x^2 + y^2)^2 + 2(x^2 + y^2) + \mathrm{E}x$

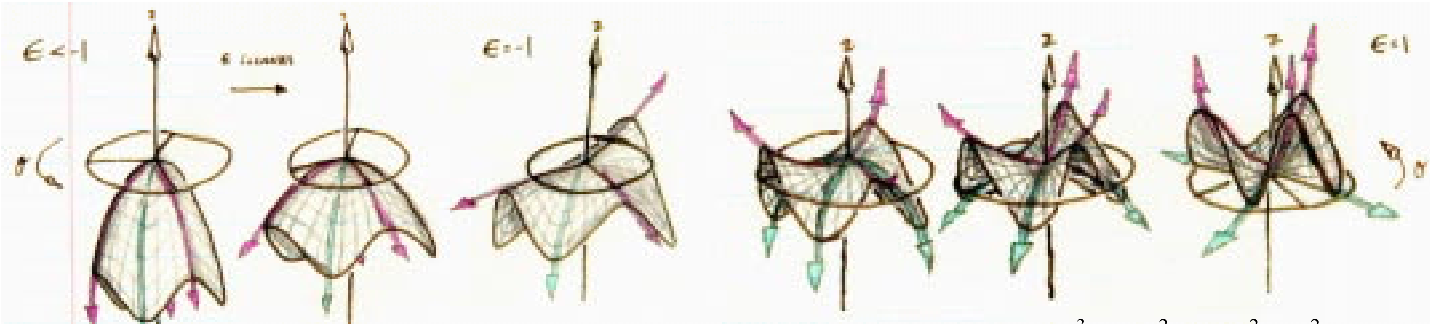
All the tools we have learned to use in representing two and three dimensions can be applied in higher dimensional settings as well. In a strictly mathematical sense, functions with four variables don't behave any differently than those with three or two. Each of the variables in a mathematical equation is equally important and *interchangeable*; no one particular variable gives the function a unique fourth axis upon which to change.

Some people find it convenient to talk about the fourth dimension as being *time*. Time is *another* dimension, but *unlike* spatial dimensions (and most mathematical variables), time is almost always conceived of as moving forward in an irreversible direction. Thus, it is unique and no longer interchangeable with the spatial dimensions. We can use the time axis to better understand certain functions & shapes, but we should not allow ourselves to see time as *the* fourth dimension.

As with the lower dimensions, our aim in drawing functions & shapes is always to communicate effectively to the viewer. So when we want to represent a complex function with four variables, we almost never attempt to draw a complete 4-D representation, which is hard enough to fathom even for the simplest 4-D objects like the hypercube.

Instead, a set of well-chosen 3-D 'slices' is far more comprehensible and often sufficient for getting a visual (and mathematical) message across to the viewer.

The series of graphs above show how *E* changes in the function $z = -(x^2 + y^2)^2 + 2(x^2 + y^2)^2 + Ex$ from a negative to a positive value. By looking at the equation and at the drawings, you might notice how this function differs from the three-variable function, $z = -(x^2 + y^2)^2 + 2(x^2 + y^2)^2$, that we considered earlier.



$z = x^3 - 3xy^2 + E(x^2 + y^2)$

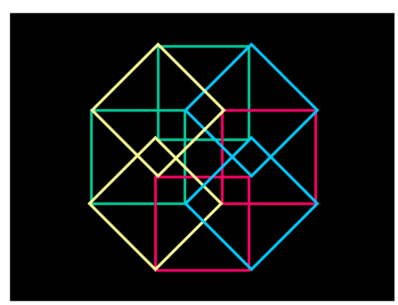
Both in the set of graphs on the previous page and in the one shown above, particular care has been taken to illustrate the most interesting changes that occur along the dimension that we are slicing. On the previous page, the function's most notable features were its critical points and how they changed as *E* changed. In the first picture the only critical point is a global maximum. As the top of the function begins tilting to form a crater, a saddle point arises, and with it a local minimum. When *E* = 0, the crater is no longer tilted; so instead of a single maximum, we now have a ring of maxima & no more saddle point. The same pattern occurs in reverse as *E* continues to increase above 0.

In the function shown above, *E* has more of an effect on the curvature of the ridges and valleys, which are highlighted in blue and red. For most values of *E*, this function has only a single critical point at the origin (although notice that the point changes from a maximum to a strange point, known to some as the *monkey saddle,* as *E* goes above -1).

Three-dimensional slices through four-dimensional objects can be useful for the illustration of certain mathematical properties, but they don't really force the mind to imagine a truly four-dimensional structure. They neglect the sense of *wholeness* in a function or object that we take for granted in 3-D renderings of 3-D objects. While it is nearly impossible for our minds to grasp the wholeness of most complex 4-D surfaces, it *is* possible to gain intuition about a certain subset of four-dimensional objects that naturally keep within simple but specific constraints.

Four-dimensional polytopes

Like the polygons & polyhedra of two & three dimensions, four-dimensional *polytopes* can be represented in ways that preserve both the symmetries and the wholeness of the object — occasionally without biasing any one of the four axes. The best example of a representation that meets all of these criteria is the *octagonal projection* of a hypercube.



This representation is equivalent to the hexagonal projection of a cube that we drew by connecting the vertices of two squares of equal size. However in order to be comprehensible & accurate, the octagonal projection requires you to draw *very* carefully or with a ruler & compass — or, as I have done on the right, with a simple computer graphics program.

What properties of a hypercube are preserved in this representation? What about in the hexagonal drawing of a cube? A hexagonal cube

has only two squares, but how many cubes are drawn in making the octagonal hypercube? And how must these cubes fit together, given the way such things are configured in three dimensions?

I find the octagonal projection to be a good esthetic representation partly because it preserves the principle of circumscribability: just as cubes & icosahedron fit within the border of a sphere, hypercubes must fit just within the border of a hypersphere. And since every 2-D slice of a sphere is circular, so too will every 2-D slice of a hypersphere be circular. So if we were to imagine a hypersphere circumscribed around the hypercube, the projection of this 4-D border would lie just outside of the octagon. Furthermore, in a circumscribed square, both points on every edge would touch the circle. In a circumscribed cube, all four vertices on every face would touch the outside sphere. Therefore, it is only reasonable to conclude that all eight vertices on every cube will touch the surrounding hypersphere in a circumscribed hypercube.

It becomes even more interesting to consider how then the cubes are fitting together. In three dimensions, every square face on a cube touches four others along its edges. For a two-dimensional creature to imagine such a configuration would be impossible, although as three-dimensional creatures we know that cube-like objects do occur. In a hypercube, every cubic 'face' must therefore have cubes on all six of its sides. Within an actual four-dimensional setting, such a configuration could occur without distorting the constituent parts — and using only eight adjoining cubes.

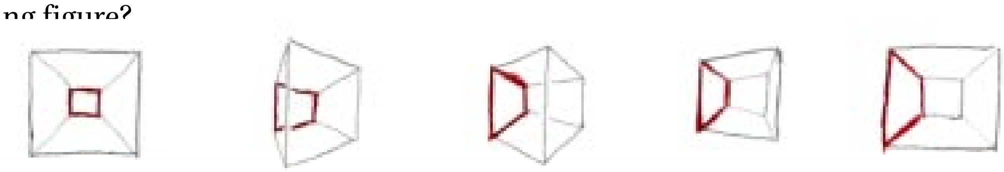
Rotating around the hypercube

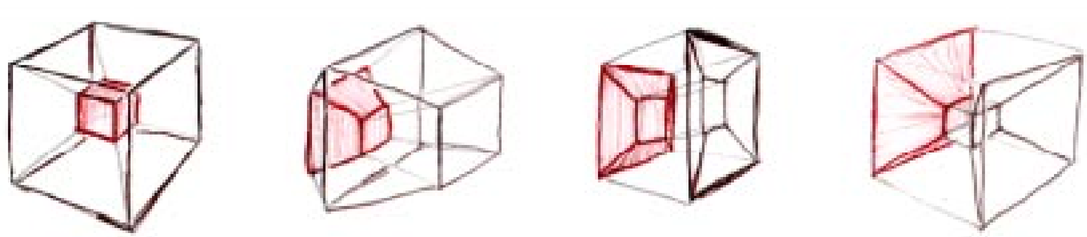
In trying to understand the geometry of a hypercube, it can be useful to draw stages that a hypercube must go through during a rotation around its 'fourth' axis. Of course, how one draws such a 4-D rotation depends on how one chooses to insert a fourth axis into the picture, and also which rules to break in the process of doing so.

The *central projection* of a hypercube (shown in the bottom left corner of the page) is analogous to looking at a transparent cube face-first (shown just above the hypercube), where perspective makes the central square *look* smaller than the one 'in front.' In some sense, depth in this head-on view extends directly toward the center of the drawing. In the same manner, the central cube *appears* smaller than the surrounding one because it is further away from the viewer along the hypercube's fourth axis.

As the cube rotates through one quarter-rotation, the smaller square (shown below in red) mutates into a trapezoid as it takes a new place on the side of the cube. Clearly, another quarter rotation and the trapezoid would mutate into the larger surrounding square.

Consider the analogous changes that occur to the smaller cube as the hypercube takes a quarter turn around its axis. Can you identify the mistake in the series below? Can you draw the correct intervening figure?





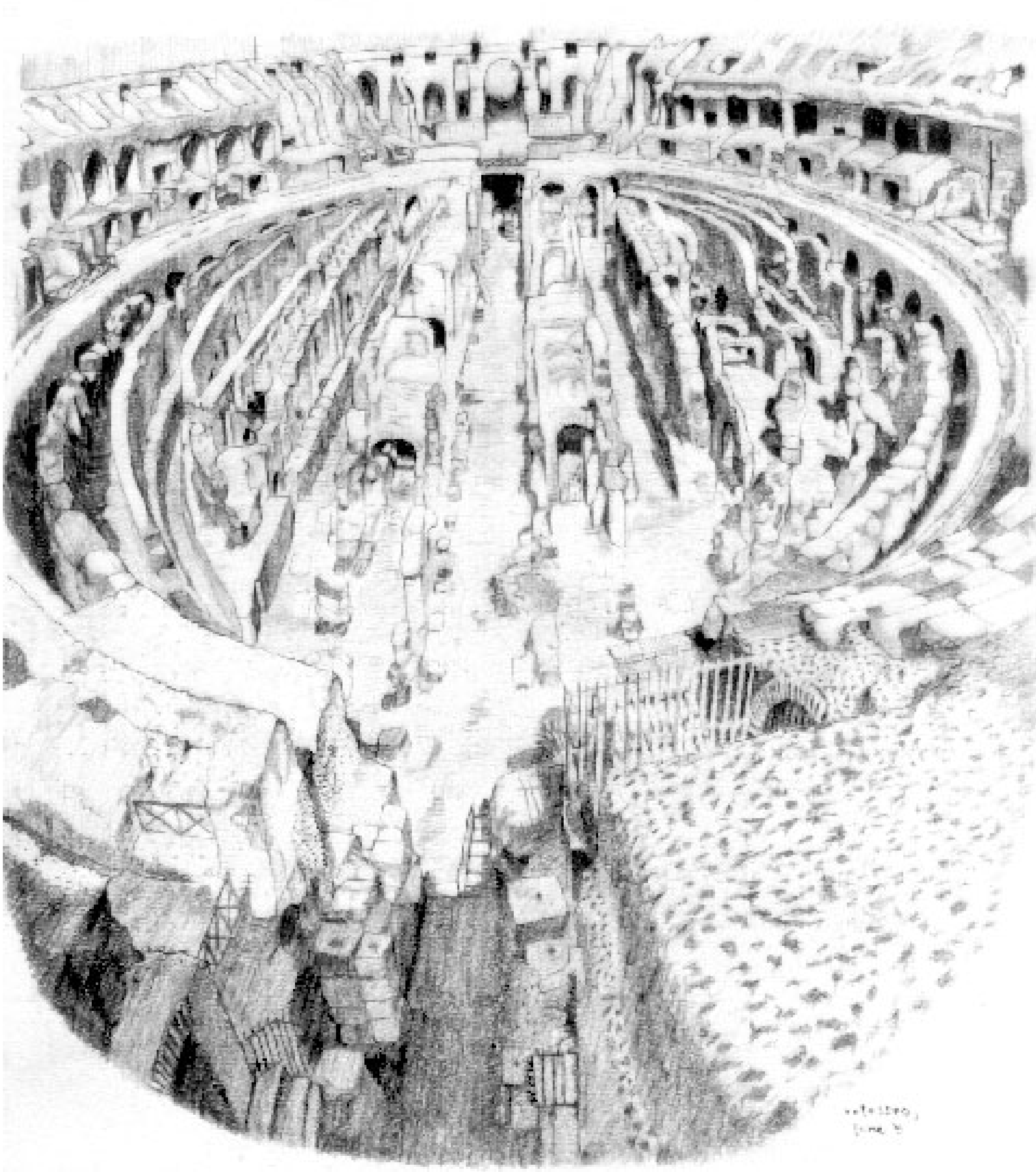
The continuing problem of *hyper-depth*

These exercises only provide an initial husk of understanding into the structure and ultimate *depth* of the fourth dimension. We can draw an octagonal figure filled with squares and call it a hypercube. We can draw strange conjunctions of objects that seem to stretch in impossible ways, and we can call that a rotating hypercube. Other modes of representation allow us to fold & unfold hyper-objects into simpler three-dimensional configurations.

But none of these schemes truly tackle the following question: where does the hypervolume, the true sense of four-dimensional depth, actually reside in these representations? Can we begin to imagine how cubes can serve as the boundaries of an object? Can we ever see a four-dimensional structure without reflexively interpreting it as a set of strangely behaving three-dimensional structures?

The brief answer to both is: yes, but only by letting go of some basic assumptions about space while looking at such figures. On an ambiguous surface where convexity or concavity seem equally possible, we must no longer assume that only one answer must be correct. In four-dimensional renderings, a surface may be both convex & concave simultaneously. In order to break free of so many perceptual habits, fundamental assumptions about the way our world behaves must be put on hold — in a manner not unlike a leap of faith.

And if you are able to take this leap, take a glimpse into a realm that had seemed unknowable, and then return & rebuild your original perceptual landscape, then you will no doubt feel as if you have discovered a new world. As you look around at the world before you, all the inferences and assumptions that had previously seemed obvious will take on a new flexibility. Patterns may emerge that had previously been ignored out of habit. And it will likely seem as if the world you have discovered is the very one from which you began the journey.



pencil drawing of the Roman Colosseum in its current state: a cylinder sliced at the level of the dressing rooms