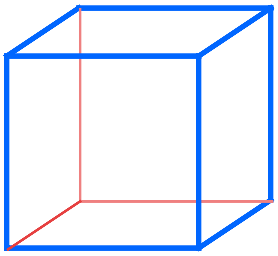
Duality

by Thomas Leavitt

The cube has 6 faces(or sides). Each side is a square. It also has 8 vertices. There are 4 vertices per face and 3 faces per vertex.



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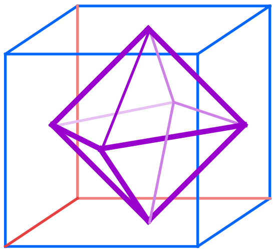
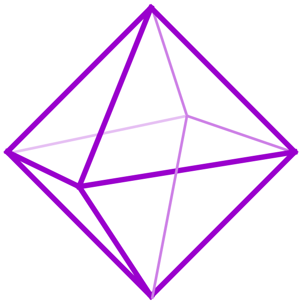
Each face has a midpoint. By connecting the midpoints of the cube's faces, we get another figure, the face dual. The cube's face dual fits inside the cube and is called an octahedron. Where there was a face there is now a vertex and vice-versa.



The octahedron has 8 faces. Each face is a triangle. It has 6 vertices. There are 3 vertices per face and 4 faces per vertex.



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These facts show the connection between the cube and octahedron:

	Vertices	Faces	V. per F.	F. per V.
Cube:	8	6	4	3
Octahedron:	6	8	3	4

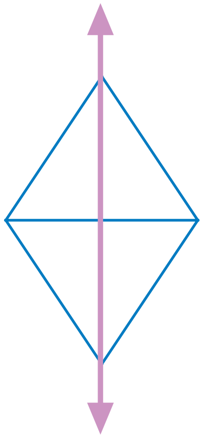
We can see here that the facts of a cube are reversed for an octahedron. The connection is called duality. So, the face dual of the cube is the octahedron. Likewise, the face dual of the octahedron is the cube.

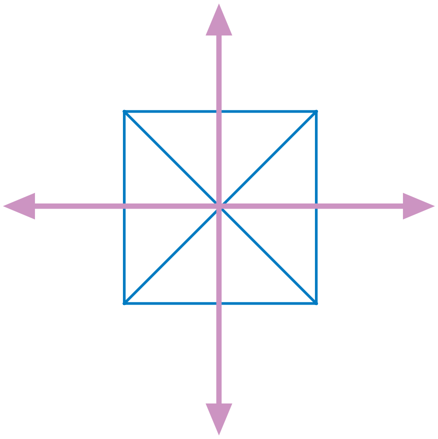
Duality is a very neat relationship. Notice that the facts in the table above work for the formula Edges+2=Faces+Vertices. For the cube, which has 12 edges, it would work out to be 12+2=6+8, or 14=14. For the octahedron, it would be 12+2=8+6, the same as the cube. This formula works for all polyhedra.

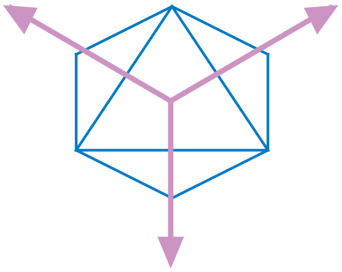
Symmetry

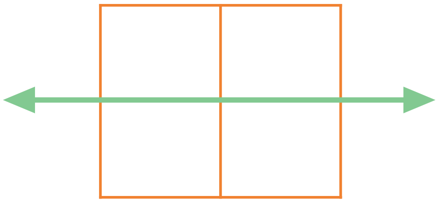
Each set of three drawings shows the same object from a different orientation. The arrows show that the object is the same in the indicated directions and hence display the symmetry of the object. For each set you are to

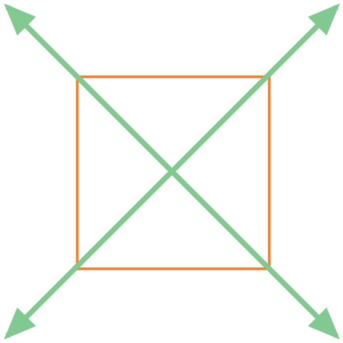
(a) Identify the object.
(b) Draw a 3D picture of the object with arrows or lines showing the direction for each displayed view.
(c) Each object has three types of axes: face to face, vertex to vertex, and edge to edge. An axis is drawn from middle to middle, so a face to face axis goes from the middle of a fact to the middle of the opposite face. Put one of each type of axis in your drawing.

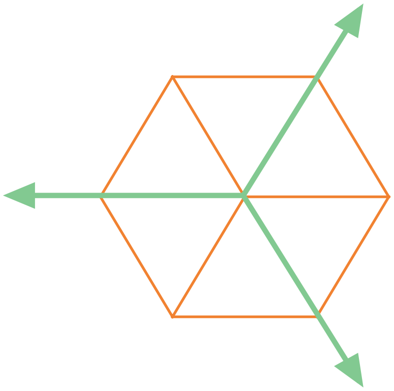








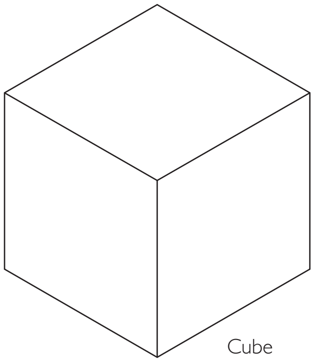




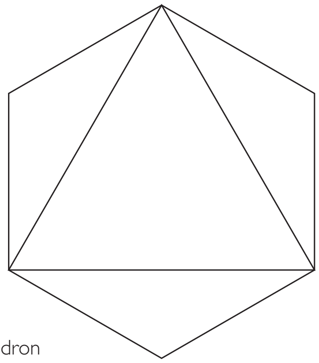
Drawings by Thomas Leavitt

Creating 3D Shapes

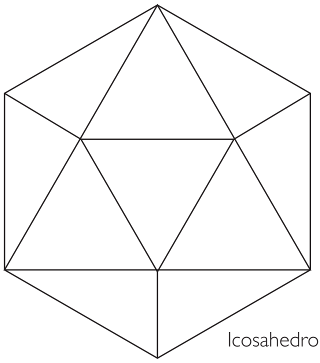
"I was working with Mike on a way to draw a particular Escher object, which at first was really hard to think about. Mike was ready to try some vector math to get the right 2D angles when I notice that all of Es cher's vertices seemed to hit the lines of a hexagon. This caught my attention because I knew that adding a few lines to a polygon often makes a picture of a polyhedron, although in perspective. For example, a hexagon can make a : icosahedron, a cube, an octahedron, and even a cubo-octahedron.



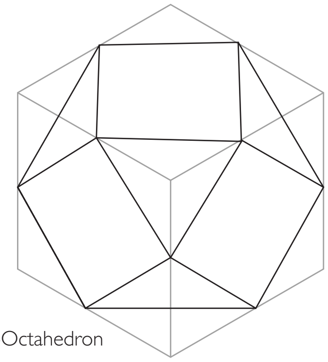
Cube



Octahedron

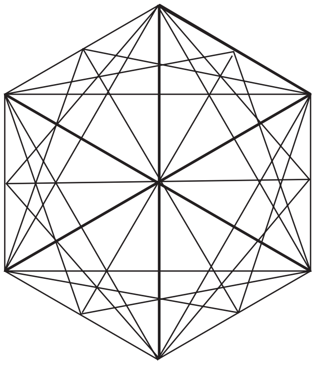


Icosahedron

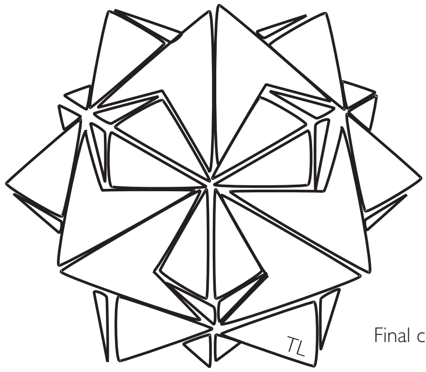


Cubo-Octahedron

Anyway the fact that the vertices lined up excited me because it meant that maybe my picture could be done the easy way. I started with a hexagon and drew lines from the midpoints to the two nearest vertices. Then I connected the midpoints of opposite edges with a segment. Then I inscribed the two equilateral triangles in the hexagon.
At this point the drawing looks as shown.



Initial sketch



Final copy

Now it's just a matter of erasing lines. This figure is great because despite its complex look, it required no angle chasing or trig at all. The hardest part is drawing a hexagon." — Thomas Leavitt

Duality

written by Andrew Berry

There is a world of geometric objects. The method of duals is a way to create new and potentially interesting ones. The easiest way to learn duality is to start with lower dimensions and to work your way to higher dimensions. The higher up in dimensions one goes, the more different types of duals can be constructed. Let's start with something simple, like a triangle. By placing a point at the midpoint of each edge, we can construct the dual object by connecting these points with lines. Each line has been replace by a point and, dually, each point with a line. In this example, however, the dual is merely another triangle. In fact, in two dimensions, this will always be the case. If we start with a shape with n sides, and connect the mid points of adjacent edges, then the resulting shape, or edge dual, will also have n sides.

After drawing a number of examples down on a piece of paper, I noticed that connecting the mid points of the line segments of the edge dual creates another shape that has become more regular. In other words, a five sided shape approaches a pentagon, a six sided shape approaches a hexagon, etc. This is an interesting tangent, and I may return to it.

Now let's jump up to three dimensions. We can now create two different duals here. As before we can connect the midpoints of adjacent edges. This edge dual method creates more and more complex shapes the higher up in dimensions we go. The other way to achieve a dual is to connect the centers of adjacent faces. This is called the face dual. Let's start first with the face dual.

When one creates face duals of regular polyhedra interesting things start happening. Let's start with the 3D extension of the triangle, the tetrahedron (A 3D extension is a 2D geometric shape "pulled" into the next dimension). The tetrahedron is the 3D extension of the triangle because first, a triangle is made of lines and a tetrahedron is made of triangles. Second, if we were to take three equally spaced points on a plane one would get a triangle. If one were to place four equally spaced points in space, one would get a tetrahedron. The duals, therefore, should also be somehow related, and they are. By connecting the centers of adjacent faces of the tetrahedron, we get another tetrahedron. It behaves in 3D the same way the triangle behaved in 2D.

So, what if we were to take a cube. Briefly hopping back to 2D, the edge dual to a square is a square. Therefore the face dual of the 3D extension of the square, the cube, must share some duality characteristics with the edge dual of the square. Our logic works again. The face dual to the cube is another regular polyhedra, the octahedron. It would be logical to next examine the octahedron's face dual. Connecting the centers of adjacent faces, we notice that the octahedron's face dual is the cube. We have stumbled upon some interesting behavior. These two regular polyhedra are the duals of each other.

	cube	octahedron
faces	6	8
vertices	8	6
intersecting faces per vertex	3	4
edges per face	4	3

We can try to establish a couple of rules using this example, which may come in useful when analyzing other shapes in 3D. In order for two shapes to be duals of each other, the number of faces in object A must equal the number of vertices in object B. And for B, vice versa. Since a vertex for object B is placed at center of each of object A's faces, the number of vertices on B equals the number of faces on A.

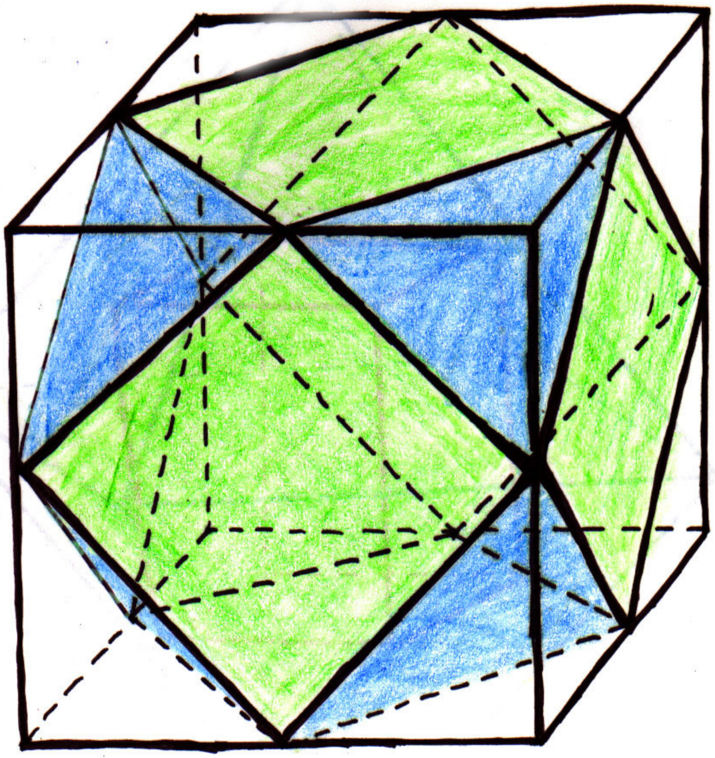
Faces of the face dual are created by "slicing" off the vertices of the original polyhedron. So, by much the same logic, a face is created where there was once a vertex.

Finally, let us look at why the edges per face on A equals the intersecting faces per vertex of B. A vertex of B is placed at the center of a face of A. The edges per face of A equals the vertices per face of A. Since a face of B replaces a vertex of A, the number of vertices per face of A equals the number of faces of B that are coming together at this particular vertex. Thus, the number of edges per face (because edges/face = vertices/face) of A equals the number of faces of B intersecting at this vertex. While all of this sounds complex, working with objects will help you to visualize what I am trying to say.

We have now established rules for duality between two 3D objects. Before we reach any premature conclusions, let us finish analyzing the two remaining regular polyhedra: the dodecahedron and the icosohedron. When we get up into more complex polyhedra, it becomes harder and harder to draw duals The dodecahedron is twelve sided and is comprised of pentagonal faces. There are 5 vertices to a face, and 12 faces to a dodecahedron. The 12 pentagons, therefore, are made up of 60 vertices. But since each vertex is shared by three pentagons, we must divide 60 by 3 and thus reach 20. A dodecahedron, therefore, has 20 vertices. By the duality rules established above we can find out the number of edges and vertices of the dual to the dodecahedron, whatever it is.

	dodecahedron	its dual
faces	12	20
vertices	20	12
intersecting faces per vertex	3	5
edges per face	5	3

We now have much information on the dodecahedron's dual. The thing that should catch your eye is that the dual has 20 sides. This is interesting because this is how many sides an icosohedron has. In order to verify this theory, we will need to get the other three relevant pieces of information. By merely looking at an icosohedron, we can see that there are 3 edges per face, and 5 intersecting faces per vertex. The last piece of info we need to get is the number of vertices. There are 3 edges per face and 20 faces per icosohedron. The 20 triangles are therefore made up of 60 vertices. But since each vertex is shared by 5 faces, the actual number of vertices is 12. Hurrah! We have proven that the dual to a dodecahedron is an icosohedron (as you may have found out from constructing it yourself). It is fascinating that the five face duals that can be constructed from the five regular polyhedra are the five regular polyhedra. This result prompts a question. If the edge dual of a regular 2D object results in that same object, and the face duals of the five polyhedra are the five polyhedra, then what results should we expect from analyzing the cell duals in 4D? I will explain the cell dual later on in this analysis. I just wanted to prompt this question now so you can dwell on it as we go more in depth into the subject of duality.



CuboOctahedron by Emily Englehardt