BEYOND 3-D *by Michael Matthews*

After one learns that in three space only five regular polyhedra exist, one might feel that there is not much else in the subject of polytopes to think about. Well it is both interesting and frustrating that this is not the case.

One of the amazing aspects of moving into higher dimensions is that you can (i) ask more questions about your objects and (ii) do more things to your objects. For example, as cubes move into successive dimensions they are accompanied by a new "perspective" or angle at which to examine them. This "new perspective" always occurs at the vertex of the cube and it presents phenomenon not encountered before (granting that the other perspectives will still be "new", but their phenomenon will have a greater correlation with a phenomenon already seen in the dimension below). Take a look at slicing cubes starting from a square (from edge to vertex), then cube (from face to edge to vertex), then 4 cube (from solid to "face" to edge to vertex) (pp. 43-49). The "new" phenomenon each time occurs w/ slices starting at the vertices.

To begin another discussion, a collection of dice may helpful when thinking about polyhedra (and just fun to have in general). The observation is that the way the number of faces of the polyhedra relates to their number of vertices is counter-intuitive: The cube (less faces) has more vertices than the octahedron, and the dodecahedron (less faces) has more vertices than the octahedron (i.e. would expect polyhedra w/ more faces to have more vertices?). More necessitates less. The observation of this phenomenon came from the fact that the cube is dual to the octahedron and that the dodecahedron is dual to the icosahedron. I assume that the phenomenon stems from the nature of the figures that the polyhedra are comprised of: squares have more sides (edges) than triangles, pentagons have more sides than triangles. Thus polyhedra created from squares/pentagons will have more vertices (have to connect more lines) than those created from triangles. How does this phenomenon of more necessitates less relate in higher dimensions (the fourth in particular)? The chapter is a bit unclear and does not present all the relevant data to this discussion.

An important difference in the fourth dimension is that a duel figure is not created by connecting adjacent "centers of faces (polygons)" like in three dimensions but instead by connecting adjacent "centers of solids (polyhedra)" (page. 98). Adjacency is determined by sharing a similar vertex. Example: a hypercube has 16 vertices, each vertex being shared by four symmetrically spaced cubes. Find center of cubes around a vertex and connect them into a regular three dimensional tetrahedron. Thus the dual to the hypercube will be composed of 16 tetrahedron. How many vertices does this 16-cell have? Eight obviously, one for each "face" (i.e. cube) of the hypercube. So the phenomenon that the number of vertices of a polytope with more "faces" than another polytope will have less vertices.

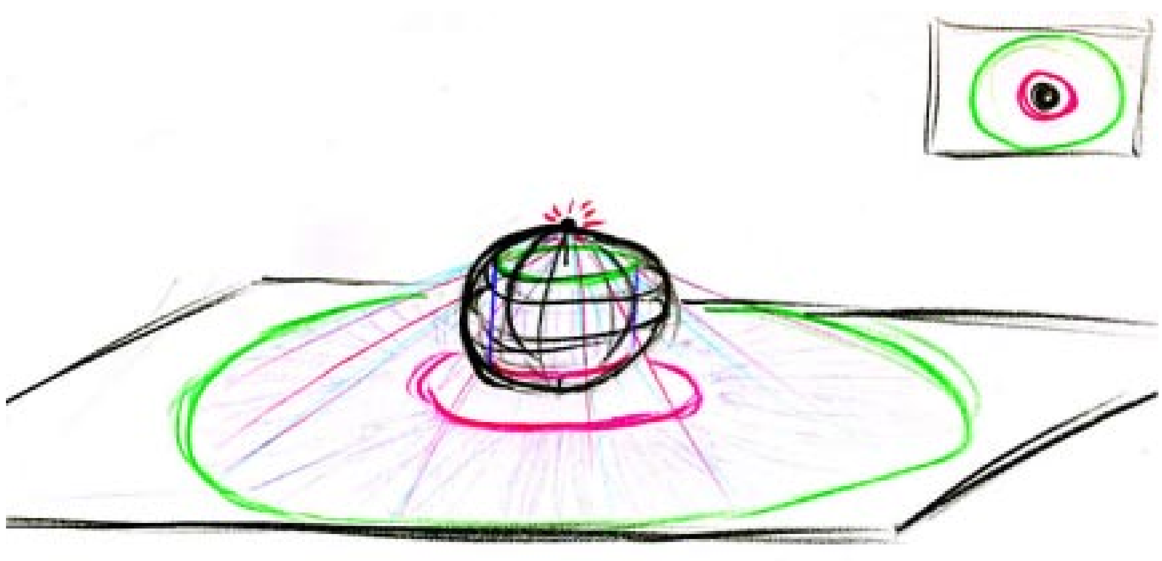
By moving our idea of duality in three space to duality in four space we come upon an interesting case of perspective. The idea of "analogous" has two different meanings: a 3 cube is the analogue of the 2 cube because it is the "square dragged parallel to itself" as presented in Flatland. By "analogy", the 4 cube is the analogue of the 3 cube because it is the three cube dragged parallel to itself. The assumption is that there is a progression of cubes (or simplexes for that matter) through the dimensions and that they are all analogous to each other. This is correct. The ambiguity in the term "analogous" arrives when you try and say that the same parts of the cubes (i.e. edges, vertices) are analogous to each other in each dimension.

Examine how the respective cubes relate to each other in different dimensions: in 2D, a line bounds the square, while in 3D a face does the bounding, in 4D a solid, etc.. What then is the analogue of a side of a square? a square is bounded by 1D objects connected at a 0D object (the vertex). In three dimensions, a cube is bounded by 2D objects connected at a 1D object. In four dimensions, a 4 cube is bounded by 3D objects connected at a 2D object, etc.. Thus the "face" of the cubes (what a n-dimensional being would see) goes (from 2D->4D) from a line to a plane to a solid. Thus the "vertex" of the cube (what the bounding objects are connected with) goes (from 2D->4D) from a point to a line segment to a square. (This pattern is the reason why William Stringham (pg.94) probably had a difficult time convincing everyone on his ideas for constructing 4D polytopes even though his ideas were technically correct: they just weren't "analogous" Euclid's construction of 3D polyhedra).

So, when we talk about "duality" in three dimensions we are talking about a phenomenon, though created in similar ways, that has a different "value" than duality does in four dimensions. In three dimensions we connect centers of faces; in four we connect centers of solids. The analogy in two dimensions would be to connect centers of sides (edges).

Back to the discussion presented first, how does the relationship between vertices and faces in three space relate to other relationships (to edges in three space, solids in four space)? This is to be put off to a later date, except for a little discussion:

Another interesting idea to think about is the similarity in how an icosahedron is built and a 600 cell is built. In three space, because I can see it, I can understand easily the two ways of building an icosahedron: i) align five equilateral triangles around a point and fold into 3 space to form a pentagonal pyramid; or ii) w/ a regular pentagon, pick a point in space and connect it to the corners of the pentagon so that the distance from the corner of the pentagon to the point in space equals the side length of the pentagon. I can see the pentagon that the triangles fold into to and can thus picture in my mind different qualities of the object. For a 600 cell though, matters are not as clear. The two methods for construction of 600 cell are: i) align 5 tetrahedron around an edge and connect the faces into four space creating a ???; or ii) w/ an icosahedron, pick a point in four space and connect it to the vertices of the icosahedron so that the distance from the vertices to the point in space equals the edge length of the icosahedron. By method i we know that around any edge of the 600 cube there will be 5 tetrahedron. By method ii we know that around any vertex there will be 20 tetrahedron. How/why/what/where does this happen? At this point it is difficult to intuit how these 4D shapes work and fit together.



STEREOGRAPHIC PROJECTION

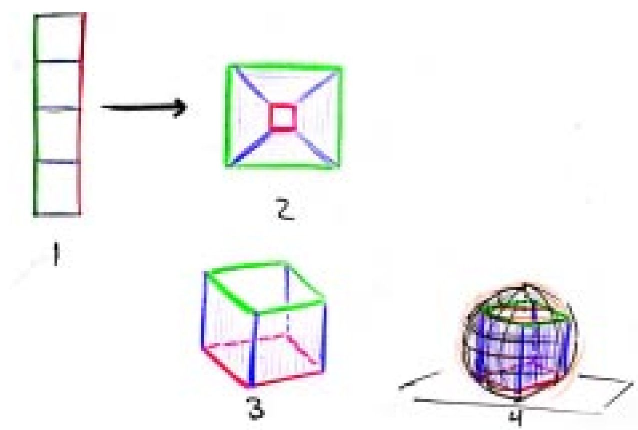
THE CLIFFORD TORUS *by Michael Matthews*

While reading chapter six, I did not feel that I had a good grasp on the Clifford torus until I examined the analogous situation for a Flatlander. In doing so I gained insight on many of the topics presented in this chapter: interrelatedness of a cube and a sphere (whatever dimensional), rotation in general (including rotating cube, rotating hypercube (both the stereographic type shown in the book and the one in the film of the 2 cubes sliding in and out of each other), rotating 2 sphere, rotating 3 sphere), stereographic projections, and cylinders and their 2 dimensional counterparts.

A FLATLANDER'S CLIFFORD TORUS

The Cube

Working totally through analogy, here is the Flatlander's equivalent to the process presented in the book. Present a Flatlander with a "wingless" unfolded cube (fig. 1), and tell him (out of simplicity, I use "him") to connect the ends. Lacking the dimensions required to connect the ends without bending the squares, the Flatlander comes up with figure 2.



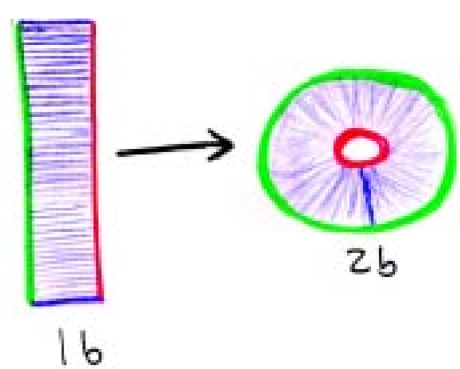
This interpretation maintains the squareness of the fold out (we see the 4 squares), but as Spacelanders we know that the intended connection would create a box with no top or bottom (fig. 3).

What is interesting about this outcome lies in the connection between figures 2 and 3: the Flatlander would actually create figure 2 when presented with the problem, but we would create the three dimensional figure 3, which would have the property that projecting it on the plane would give figure 2. Thus our figure 2 is both a 2D creation and a stereographic projection of the 3D cube.

Furthermore, the sections in the chapter describing the Clifford torus pointed out that the 16 vertices of the hypercube all resided on a hypersphere (3 sphere), and analogously, the 8 vertices of our box reside on a 2 sphere (fig. 4).

The Cylinder

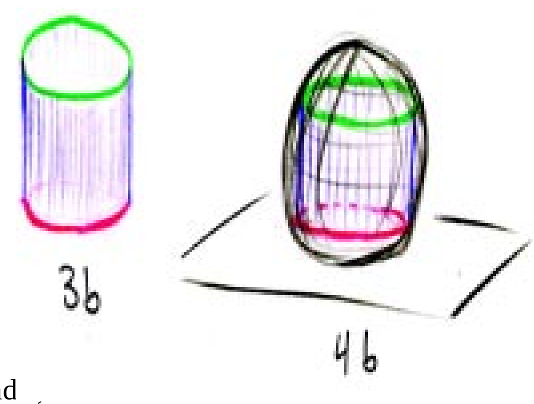
To make things cleaner, we can divide our "wingless" cube fold out into lots of thin slivers (fig. 1b), and then repeat our process with the Flatlander. When he connects the edges this time the outcome is a simple ring (fig. 2b).



Notice how the Flatlander had to stretch the green spine while constricting the red spine (note connections w/ situation in fig. 2). We as Spacelanders know that the figure can be connected in three dimensions in such a way that nothing is stretched, and this construction gives a cylinder (fig. 3b).

[Once again we see how the Flatlander's construction is related to our 3D construction. Not only is the ring the Flatlander's construction, it is also the stereographic projection of our 3D cylinder.

Furthermore, as the division in the unfolded cube goes to infinity, the constructions become closer and closer to being a true ring and a true cylinder. Likewise, as every division is made, more vertices on the cylinder are created, each vertex lying on the two sphere. When a true cylinder is reached (effectively reached), we will have the situation presented in figure 4b, a cylinder inscribed in a 2 sphere (just like fig. 4). A phenomenon unique to the cylinder, however, is that every point on the green and red rims lies on the sphere, effectively producing the situation similar to page 125 in B3D of a sphere with rings traced around it. Thus, when we project stereographically, we are projecting not just a cylind but a ringed sphere.

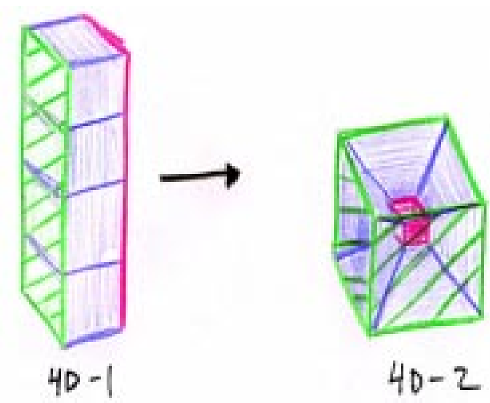


This ring is the Flatlander's Clifford torus.

Our Clifford Torus

The HyperCube

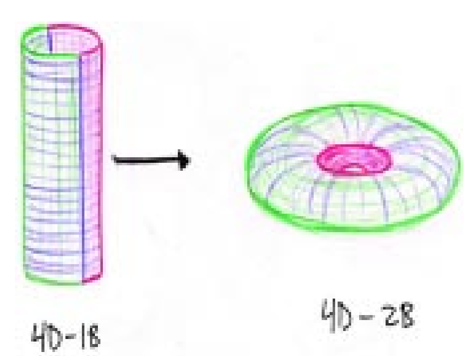
The analogous situation to the above in three space was presented in B3D. If we were presented with a "wingless" fold out of a hypercube (fig. 4D-1), and told to connect the rims, the best thing we would be able to come up with would be figure 4D-2.



We lack the dimensions needed to connect the rims without bending the cubes; we would need at least four dimensions. We know that in four dimensions, the folded up figure would be a hypercube and would be inscribed in a hypersphere (the analogues to figures 3 and 4). The crucial point: the object we created out of figure 4D-1 in three space is the same thing as the stereographic projection of what a 4 dimensional being would create out of figure 4D-1. Figure 4D-2 is both a creation and a projection.

The HyperSphere

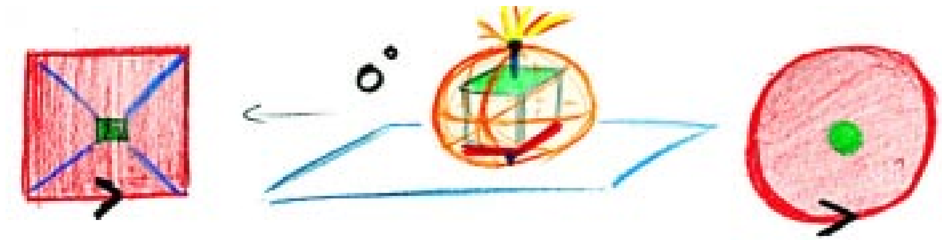
To make things cleaner, we can divide our "wingless" hypercube fold out into thin-many-sided-polygonal-prisms (a cylinder, fig 4D-1B) just as we divided the cube fold out into thin slivers (we have to divide the cube up in two ways, giving the 8x8 grid on page 126, rather than just one way like in the cube because we have an extra dimension to deal with). Now when we connect the rims in three space, we get figure 4D-2B, which looks just like our normal torus.

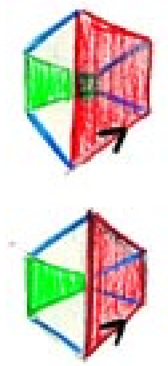


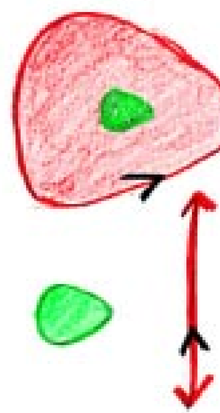
Notice how the green spine is stretched and the red spine is constricted. We can't imagine the construction, but a four dimensional being would be able to connect the rims of the cylinder so nothing stretched (analogue of fig 3b). Just like the rims of our earlier cylinder traced out two circles on the circumscribed 2 sphere, the "rims" of our "hyper-cylinder", which are two dimensional (the green and red faces on figure 4D-1B), will trace out two higher dimensional analogues to the circles before, a red and a green 2 sphere. Thus the stereographic projection from four space to three space of our hypersphere (or hyper-cylinder) with marked red and green "rings" will be the Clifford torus.

Thus, the torus that we created in three dimensions by "bending" figure 4D-1B is the same the Clifford torus (the stereographic projection of what a 4 dimensional being would create). They are the same thing but different at the same time, just as the rings for a Flatlander are the same thing but different. This was all very confusing until I traced out the dimensional analogue.

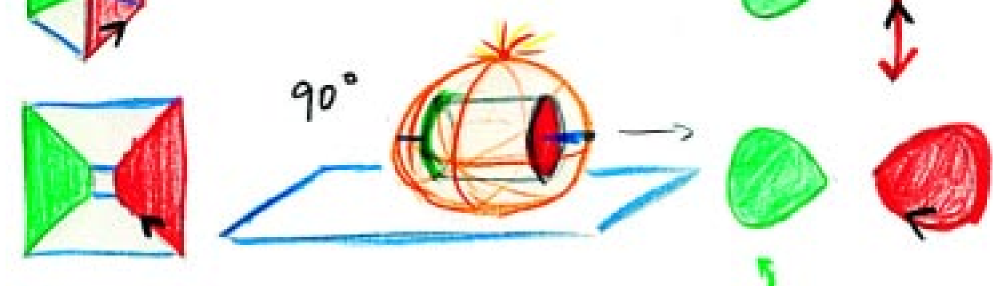
Rotation

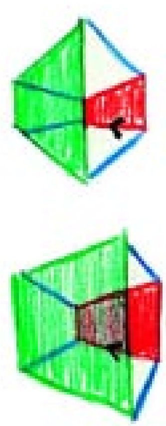


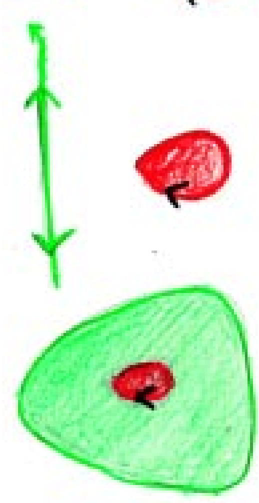




Just like the rotations in four dimensions of the stereographic projection of the hypercube and the Clifford torus look similar, the projections of the cube rotating and the "cylinder" rotating are interrelated and similar as well. Here we will give the light source on the sphere (red dot), the objects inscribed into their spheres, the angle of revolution, and the projection. Each object has a green face and a blue face. It is very effective







to follow a single column all the way down. Notice two things specifically: 1) how closely related the two projections are (and why they are different); 2) how related the two projections are to the projections of the rotating hypercube and 3 sphere. The blue arrows on the green curves are to better show the "orientation" aspect of the rotations, and how, to a Flatlander, the curves would seem to turn inside out (right handed becoming left handed).